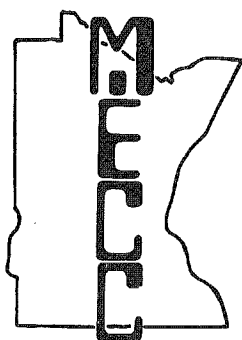
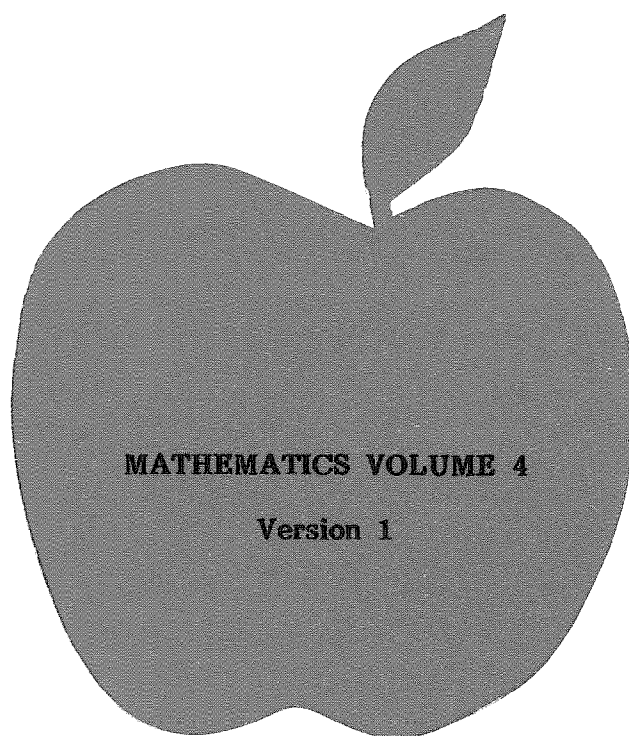


1 copy



MINNESOTA
EDUCATIONAL
COMPUTING
CONSORTIUM



MATHEMATICS VOLUME 4

Version 1

FOR USE WITH THE APPLE II™ MICROCOMPUTER

MATHEMATICS VOLUME 4

Version 1

TABLE OF CONTENTS

Introduction	1
Acknowledgements	
Index to Programs on Diskette	2
General Description	3
Use in an Instructional Setting	4
Handout #1 - Mathematical Expressions in BASIC	7
Programs	
Calculus Programs	
GRAPH	8
Background Information	10
Use in an Instructional Setting	13
Handouts	
#2 - GRAPH Instructions	14
#3 - GRAPH Worked Examples	18
#4 - Cartesian Coordinate Graphing	21
#5 - Polar Coordinate Graphing	22
#6 - Graphing Parametric Equations	23
Sample Runs	24
FUNCTION LIMITS	26
Background Information	27
Use in an Instructional Setting	28
Handouts	
#7 - FUNCTION LIMITS Instructions	29
#8 - FUNCTION LIMITS Worked Examples	30
#9 - FUNCTION LIMITS Applications	31
#9 - FUNCTION LIMITS Answers	32
Sample Runs	33
POLYROOT	34
Background Information	35
Use in an Instructional Setting	37
Handouts	
#10 - POLYROOT Instructions	38
#11 - POLYROOT Worked Example	39
#12 - POLYROOT Applications	40
#12 - POLYROOT Answers	41
Sample Runs	42
NEWTON 1	43
Background Information	44
Use in an Instructional Setting	46
Handouts	
#13 - NEWTON 1 Instructions	47
#14 - NEWTON 1 Worked Example	48
#15 - NEWTON 1 Applications	49
#15 - NEWTON 1 Answers	50
Sample Runs	51

TABLE OF CONTENTS (Continued)

NEWTON 2	52
Background Information	53
Use in an Instructional Setting	55
Handouts	
#16 - NEWTON 2 Instructions	56
#17 - NEWTON 2 Worked Example	57
#18 - NEWTON 2 Applications	58
#18 - NEWTON 2 Answers	59
Sample Runs	60
INTEGRATION	61
Background Information	62
Use in an Instructional Setting	65
Handouts	
#19 - INTEGRATION Instructions	66
#20 - INTEGRATION Worked Example	67
#21 - INTEGRATION Applications	68
#21 - INTEGRATION Answers	69
Sample Runs	70
MULTI-INTEGRALS	72
Background Information	73
Use in an Instructional Setting	75
Handouts	
#22 - MULTI-INTEGRALS Instructions	76
#23 - MULTI-INTEGRALS Worked Example	77
#24 - MULTI-INTEGRALS Applications	78
#24 - MULTI-INTEGRALS Answers	79
Sample Runs	80
LEAST SQUARES	82
Background Information	83
Use in an Instructional Setting	87
Handouts	
#25 - LEAST SQUARES Instructions	88
#26 - LEAST SQUARES Worked Example	89
#27 - LEAST SQUARES Applications	91
#27 - LEAST SQUARES Answers	92
Sample Runs	94
Linear Algebra Programs	
LINE GRAPH	97
Background Information	98
Use in an Instructional Setting	99
Handouts	
#28 - LINE GRAPH Instructions	100
#29 - LINE GRAPH Worked Example	101
#30 - Graphing Linear Equations	102
Sample Runs	103

TABLE OF CONTENTS (Continued)

ROW REDUCTION	105
Background Information	106
Use in an Instructional Setting	107
Handouts	
#31 - ROW REDUCTION Instructions	108
#32 - ROW REDUCTION Worked Example	109
#33 - ROW REDUCTION Applications	110
#33 - ROW REDUCTION Answers	111
Sample Runs	112
MATRIX	115
Background Information	116
Use in an Instructional Setting	119
Handouts	
#34 - MATRIX Instructions	120
#35 - MATRIX Worked Example	125
#36 - MATRIX Applications	129
#36 - MATRIX Answers	131
Sample Runs	132
LINEAR PROGRAM 1	135
Background Information	136
Use in an Instructional Setting	138
Handouts	
#37 - LINEAR PROGRAM 1 Instructions	139
#38 - LINEAR PROGRAM 1 Worked Example	140
#39 - LINEAR PROGRAM 1 Applications	142
#39 - LINEAR PROGRAM 1 Answers	143
Sample Runs	144
LINEAR PROGRAM 2	146
Background Information	147
Use in an Instructional Setting	149
Handouts	
#40 - LINEAR PROGRAM 2 Instructions	150
#41 - LINEAR PROGRAM 2 Worked Example	151
#42 - LINEAR PROGRAM 2 Applications	153
#42 - LINEAR PROGRAM 2 Answers	154
Sample Runs	155
Appendixes	157
A. Getting to Know Your Apple	158
B. Working with the Apple	160
C. Definition of Terms	161
D. Credits	163
E. Technical Information	164
F. MECC Instructional Services Activities	166
G. Evaluation Sheet	167

INTRODUCTION

The module, Mathematics Volume 4, consists of this manual and a diskette containing thirteen computer programs. It is designed for use with college and university mathematics courses.

The programs provide a variety of mathematical tools in the areas of calculus and analytic geometry, linear algebra, and finite mathematics. By using the computer to quickly perform otherwise tedious and time-consuming numerical calculations, students can study more complicated, realistic problems. They can concentrate on the analysis of the problem and the interpretation of the results, rather than the intermediate calculations.

These programs can be an integral part of standard mathematics courses, such as calculus or linear algebra. Students would be assigned problems to be solved using the computer to perform the necessary calculations. Alternately, the programs might form the basis of a separate course in computer applications in mathematics, possibly as a lab course in conjunction with standard math courses.

It is suggested that a printer be used in conjunction with many of these programs. Detailed instructions for the Printer Option are found in the Use in an Instructional Setting portion of this manual.

Students using these materials do not need to know computer programming. After a brief introduction to the Apple II microcomputer and a demonstration of one or more programs, they should be able to run these programs without assistance. Instruction sheets and worked examples are provided for each program to help insure the students' success.

Handout pages in this manual may be duplicated for use with students. These pages are numbered sequentially in the upper right hand corner, for example, Handout #1 - Name of Handout.

ACKNOWLEDGEMENTS

This manual was edited by Lois Edwards, MECC, from documentation written by Beverly Durkee, Augsburg College, Minneapolis, MN. The computer programs were developed by Beverly Durkee, Augsburg College, with modifications by MECC staff (see Appendix D). Significant review comments and suggestions for this module were received from Elmer Mattila, MECC Community College Coordinator, Shirley Beil, Normandale Community College, Allen Hesse, Rochester Community College, Everett Tollerud, Joyce Tollerud, John Kronholm and Casey Humphreys, Minneapolis Community College, and Jane Chatterjee, consultant. This module is a product of MECC Instructional Services.

INDEX TO PROGRAMS ON DISKETTE

CALCULUS

GRAPH

plots curves in Cartesian or polar coordinates using equations in standard or parametric form.

FUNCTION LIMITS

investigates the limit of a function of two variables.

POLYROOT

calculates real and complex roots of any real polynomial degree 32 or less, using Newton-Raphson iterative techniques.

NEWTON1

uses Newton's method to calculate a single approximation to a root of an equation of the form $f(x) = 0$, where f is differentiable.

NEWTON2

uses Newton's method of finding successive approximations to a root of an equation $f(x) = 0$ where f is differentiable.

INTEGRATION

calculates various numerical approximations of definite integral using Riemann sums, the trapezoidal rule, and the parabolic rule.

MULTI-INTEGRALS

calculates a numerical approximation to a double or triple integral.

LEAST SQUARES

approximates a set of data pairs by a power, exponential, or polynomial function.

LINEAR ALGEBRA

LINE GRAPH

plots up to 10 lines on a single screen.

ROW REDUCTION

performs row operations on a matrix.

MATRIX

performs operations on matrices of size 10 by 10 or smaller.

LINEAR PROGRAM1

solves linear programs using the simplex method. The user selects pivot row and pivot column.

LINEAR PROGRAM2

solves linear programs using the simplex method.

GENERAL DESCRIPTION

The thirteen programs in Mathematics Volume 4 fall into two broad categories, calculus and linear algebra, however, they can be effectively used in a variety of mathematics courses, as the following suggestions indicate.

<u>Subject area</u>	<u>Programs</u>
Analytical geometry	LINE GRAPH GRAPH POLYROOT NEWTON 1 NEWTON 2
Calculus	GRAPH FUNCTION LIMITS POLYROOT NEWTON 1 NEWTON 2 INTEGRATION MULTI-INTEGRALS LEAST SQUARES
College algebra and trigonometry	LINE GRAPH GRAPH POLYROOT NEWTON 2
Finite mathematics	LINE GRAPH LEAST SQUARES ROW REDUCTION MATRIX LINEAR PROGRAM 1 LINEAR PROGRAM 2
Linear algebra	ROW REDUCTION MATRIX LINEAR PROGRAM 1 LINEAR PROGRAM 2 LEAST SQUARES
Numerical analysis	LEAST SQUARES NEWTON 2
Statistics	LEAST SQUARES
Operations research	LINEAR PROGRAM 1 LINEAR PROGRAM 2
Advanced high school mathematics	LINE GRAPH GRAPH POLYROOT INTEGRATION

USE IN AN INSTRUCTIONAL SETTING ...

Preparation

The instructor must prepare the students to use these programs by presenting appropriate background material. These programs do not teach concepts, rather they are tools to be used by students in exploring mathematical problems.

Students should also be given a demonstration of the program they will use and printed copies of the instructions and worked examples included in this manual as handouts #1 through #42.

Using the Programs

USE OF BASIC NOTATION

In most of the programs relating to calculus, the equations and expressions must be entered using BASIC notation. These programs are GRAPH, FUNCTION LIMIT, NEWTON 1 and 2, INTEGRATION, and MULTI-INTEGRALS. The operation symbols and mathematical functions used in BASIC are listed on handout #1 - Mathematical Expressions in BASIC, which should be made available to students using these programs.

PRINTER OPTION

The value of many of these programs is increased by the use of a printer connected to the microcomputer. Using a printer to record output can save students considerable amounts of time and reduce the chance of copying errors. A printed record of the actual running of a program can help the instructor discover any mistakes or difficulties students may have experienced.

Programs which produce a large amount of output or which store significant amounts of data should be used in conjunction with a printer. These programs include ROW REDUCTION, MATRIX, LINEAR PROGRAM 1 and 2, and LEAST SQUARES. The programs GRAPH and LINE GRAPH, which produce only graphics, not text, cannot be used with a printer.

Technical Information

In the Printer Option, the instructor specifies the type of card (interface) and the printer parameters for the particular printer being used. This information is stored on the diskette for use with any of the programs.

USE IN AN INSTRUCTIONAL SETTING (Continued)...

When an individual program (except GRAPH and LINE GRAPH) is being run, the user, i.e., the student, is asked whether the printer will be used. Thus, the student can request printed output without knowing the details of the printer system.

The Printer Option will print the program on both the screen and the printer. The student may work from the screen and use the printed copy as a record of the completed assignment.

The method used in this printer option is effective only if the user has a parallel card or a communications card for the printer. A serial card does not allow printing on both the screen and the printer.

NOTE: When the Printer Option is activated, the printing on the screen may slow down.

Procedure To Set-up Or Change Printer Option

1. When the computer asks "WHICH PROGRAM DO YOU WANT?" on the menu screen, type PR (for printer) instead of typing a number.
2. After a set of instructions appears, the user is asked whether the card connected to the printer is a
 - 1) communications card
 - 2) parallel card
 - 3) other type of card.

Type 1, 2, or 3.

3. If a communications card is specified, enter the printer speed (either 10 or 30 c.p.s., characters per second).

Then enter the printer width (40, 80, 132 characters)

The program will automatically return to the menu screen at this point.

USE IN AN INSTRUCTIONAL SETTING (Continued)...

4. If a parallel or other type of card is used, then enter the commands necessary to activate the printer. A Control-D (holding down the CTRL key while pressing the D key) is always needed before the PR# statement. The Control-D will appear on the screen as a D in inverse mode (black letter on white background).

Examples:

	<u>Commands</u>	<u>Appearance On Screen</u>
A printer connected in slot 1	Control-D PR#1	<input type="checkbox"/> PR#1
	Control-I 80N	<input type="checkbox"/> 80N
A printer that is connected in slot 4	Control-D PR#4	<input type="checkbox"/> PR#4
To return to using only the screen	Control D-PR#0	<input type="checkbox"/> PR#0

The program will automatically return to the menu at this point.

MATHEMATICAL EXPRESSIONS IN BASIC

The following operators are used in BASIC:

- + addition
- subtraction
- * multiplication (5x must be entered as 5*X)
- / division ($\frac{x}{x-3}$ is entered as X/(X-3))
- ^ exponentiation (x^2 is entered as X^2)

Parentheses may be used to override the normal order of operations. The standard order of operations is:

- ^ exponential
- * / multiplication and division from left to right
- + - addition and subtraction from left to right

Several built-in mathematical functions are also available. Functions are always evaluated before any operations are performed. The functions available are as follows:

SIN	sine	SGN	sign
COS	cosine	ABS	absolute value
TAN	tangent	SQR	square root
ATN	arctangent	EXP	exponential
INT	greatest integer	LOG	natural logarithm
RND	random number		

EXAMPLES

<u>Mathematical Expression</u>	<u>BASIC Equivalent</u>
$y = 3x^2 - 5x + 1$	<code>Y=3*X^2-5*X+1</code>
$f(x,y) = \frac{x+y}{x^2+y^2}$	<code>F(X,Y)=(X+Y)/(X^2+Y^2)</code>
$r = \sin(\theta)\cos(\theta)$	<code>R=SIN(T)*COS(T)</code>
$x = t + 1$ $y = \sqrt{t}$	<code>X=T+1</code> <code>Y=SQR(T)</code>
$y = e^x$	<code>Y=EXP(X)</code>

GRAPHING IN CARTESIAN OR POLAR SYSTEMS OR PARAMETRIC FORM _____

Specific Topic: College algebra, analytic geometry, trigonometry

Type: Problem Solving

Reading Level: 9-10 (Dale-Chall)

Grade Level: 11 - Higher Education

DESCRIPTION...

This program displays the graphs of equations expressed in the Cartesian system, in the polar system, or in parametric form. It combines the programs POLAR and POLYGRAPH from Mathematics Volume 1, and adds the option of entering equations in parametric form.

Several curves can be plotted on the same screen. The intervals shown on the screen can be changed.

OBJECTIVES...

1. to explore Cartesian coordinates by graphing with various scales, specifying maximums and minimums, and expanding or shrinking distances and sections of the graphs.
2. to analyze graphs of quadratics and their roots.
3. to investigate the graphs of conics.
4. to discover properties of trigonometric functions, their limits, their continuity, and their periodicity in cartesian and polar forms.
5. to study the relationship between polar and cartesian (rectangular) coordinates through graphing of trigonometric functions.
6. to calculate distance and minimum and maximum concepts in terms of polar coordinates by graphical methods.

G R A P H

OBJECTIVES (Continued)...

7. to visualize how Taylor polynomials approximate a given function.
8. to study geometric figures which are expressed in parametric equations.
9. to observe art in mathematics as it occurs in the graphs of equations.

BACKGROUND INFORMATION...Mathematical Background

Using the Cartesian (rectangular) system, this program graphs equations of the form.

$$y = (\text{an expression in } x) \text{ or } y^2 = (\text{an expression in } x)$$

Using the polar system, this program graphs equation of the form

$$r = (\text{an expression in } \Theta) \text{ or } r^2 = (\text{an expression in } \Theta),$$

where Θ is the Greek letter theta. Theta, Θ is represented by the letter T on the computer keyboard.

Any curve expressed as $y^2 = (\text{expression in } x)$ or $r^2 = (\text{expression in } \Theta)$ can be considered as two functions

$$\begin{array}{l} y = + \sqrt{\text{expression in } x} \quad \text{and } y = - \sqrt{\text{expression in } x} \\ \text{or } r = + \sqrt{\text{expression in } \Theta} \quad \text{and } r = - \sqrt{\text{expression in } \Theta} \end{array}$$

Parametric equations are of the form $x=f(t)$, $y=g(t)$ for functions f and g of the parameter t .

A curve in either the Cartesian or polar system can be expressed in parametric form.

For $y = f(x)$, set $x = t$ and let $y = f(t)$.

For $r = f(\Theta)$, set $x = f(t)\cos(t)$, and $y = f(t)\sin(t)$.

The Computer Model

The program is composed of four main parts:

- (1) selecting the plotting parameters
- (2) specifying the number of points plotted
- (3) entering the equation and viewing the resulting graph
- (4) exploring the graph by changing the parameters to view different portions of the graph

BACKGROUND INFORMATION (Continued)...**1. Selecting the plotting parameters****a. Cartesian Coordinates**

The user must enter the x-interval (horizontal axis minimum and maximum values). Then the user must either enter the y-interval (vertical axis minimum and maximum) or request the computer to determine an appropriate interval.

If equal spacing is requested, the user must enter a "mean y value" (the y value that will appear in the middle of the screen). Often the mean y value would be zero. The computer then sets an interval with the mean y value as midpoint.

b. Polar Coordinates

The user may request the "default theta range," which is $(0, 2\pi)$ or 0 to 360 degrees. Alternately, the user can specify the lower and upper limit (in degrees).

Then the user can either have the computer determine the axes (intervals) or can specify the x and y intervals desired. If the program computes the intervals, the user must enter a maximum radius, and the graph's origin will be at the center of the screen.

If the user specifies the x and y intervals, then the user must also specify whether or not to have equal spacing and what the mean y value should be.

c. Parametric Form

The user must enter the lower and upper limit of the parameter, as well as the interval information about x and y intervals. Use the variable T for the parameter in both expressions.

2. Specifying the number of points plotted

The user specifies the number of points to be plotted. A larger number of points results in a smoother graph but requires more computer time.

Some graphs, such as $y^2=1-x^2$, may appear to have gaps near the x axis, unless a large number of points are plotted.

BACKGROUND INFORMATION (Continued)...

3. Entering the equation(s)

All equations entered into GRAPH must be in a form the program can read, using BASIC operators and functions. See handout # 1 - Mathematical Expressions in BASIC.

4. Exploring the graph by changing parameters

After the initial graph is plotted, the user can regraph the graph for various areas as follows:

U - Up (above),

D - Down (below),

R - Right,

L - Left,

S - Shrink, (zoom in), the area in the center of the current screen,

E - Expand, (zoom away), the area including the screen and immediately surrounding the screen.

If several equations are graphed on one screen and one of these six options is selected, only the last equation plotted will appear on the new graph.

When the Space Bar is pressed, the user is given the option of creating another graph by using the same equation on new axes or with a different number of plot points, or plotting a new equation on the same axes.

USE IN AN INSTRUCTIONAL SETTING...

This program can be used in any situation where a graphical representation of a curve would be useful in solving a problem.

For example:

1. When studying Taylor polynomials, it is helpful to see the computer representation of the function and several Taylor polynomials. This provides an intuitive understanding of how the Taylor polynomials approximate the function closer for higher degree polynomials.

e.g., graph $y = \sin(x)$, $y = x$, $y = x - 1/6 x^3$, and

$$y = x - 1/6 x^3 + 1/120 x^5$$

on the x-interval $(-3.5, 3.5)$ and y-interval $(-3.5, 3.5)$.

2. Several trigonometric functions can be compared graphically to explore the effect of changes in constants.

e.g. graph $y = \sin(x)$, $y = \sin(2x)$, $y = 2\sin(x)$, and
 $y = 2 + \sin(x)$

on the x-interval $(0, 6.5)$ and y-interval $(-2, 4)$.

3. When studying integration in polar coordinates, intersections can be estimated to obtain limits of integration.

e.g. find the intersection of $r = 1 + \cos(\Theta)$ and $r = 2\sin(\Theta)$.
 Graph the two functions and estimate the intersection.

4. The study of trochoids, hypocycloids, and epicycloids requires consideration of parametric equations.

e.g., graph $x = 6\cos(\Theta) - \cos(6\Theta)$
 $y = 6\sin(\Theta) - \sin(6\Theta)$

This is a 5 cusped epicycloid.

GRAPH INSTRUCTIONS

GRAPH is a program designed to plot relations given in these three forms:

CARTESIAN

$$Y = F(X) \quad \text{-or-} \quad Y^2 = \dots \quad \text{or}$$

POLAR

$$R = F(T) \quad \text{-or-} \quad R^2 = \dots \quad \text{or}$$

PARAMETRIC

$$X = F(T) \text{ AND } Y = G(T)$$

CARTESIAN EQUATIONS must be of the form

$$Y = \text{expression}$$

-or-

$$Y^2 = \text{expression}$$

X must be the independent variable used in the expression, and
Y the dependent variable.

POLAR EQUATIONS must be the form

$$R = \text{expression}$$

-or-

$$R^2 = \text{expression}$$

T (theta) must be the independent variable in the expression, and
R the dependent variable.

PARAMETRIC EQUATIONS must be of the form

$$X = \text{expression}$$

-and-

$$Y = \text{expression}$$

(both equations are required)

T must be the parameter variable in both expressions

X must be the dependent variable in the first equation, and
Y the dependent variable in the second equation.

After a relation is plotted, one of 6 options concerning the screen window may be selected:

- U - plot region above it (up)
- D - plot region below it (down)
- L - plot region left of it
- R - plot region right of it
- S - shrink (zoom in)
- E - expand (zoom away)

If several equations are graphed on one screen and one of the 6 options is selected only the last equation plotted will appear on the new graph.

Another graph can be plotted reusing the same equation on new axes.

A new equation can be plotted over the current graph using the same axes (i.e. using the current screen).

The new equation can be Cartesian, polar, or parametric regardless of the first equation.

GRAPH INSTRUCTIONS (Page 2)

Cartesian Coordinates

- Step 1. Enter graph type - Cartesian (option 1)
- Step 2. Enter minimum horizontal x value, like -10, -1, or 0.
Enter maximum horizontal x value like +10, +1, etc.
- Step 3. Decide whether the computer will compute the y limits (the minimum and maximum y values). You may wish to respond 'yes' on your first try. Otherwise specify the y values to be used.
- Step 4. Specify whether you want equal spacing, that is, whether the interval length on the x and y axes should be the same. In most cases you will want equal spacing for your first try.
- Step 5. Enter the mean y value, that is, the value of y that will appear in the middle of the screen. You may wish to use zero in most cases.
- Step 6. Enter the number of plot points. A larger number takes longer to plot, but gives a smoother, more accurate graph. Try 100 points.
- Step 7. Enter the equation you wish to graph. Remember to use BASIC notation. ($y=x^2$ as $Y=X^2$, $y=3x$ as $Y=3*X$)
- Step 8. After the graph appears, decide whether to look at a portion of the graph using the options U(up), D(down), L(left), R(right), S(shrink), E(expand).
- Step 9. If you press the Space Bar, you may choose to plot this same graph on different axes or with a different number of points, or plot another graph on the same axes.

EXAMPLES

Acceptable Input

$$Y = 3*X^2+15*X-9$$

$$Y^{\wedge}2 = -X/3$$

$$Y = X^{\wedge}(-.25)$$

$$Y = 5*EXP(X)$$

$$Y = SIN(X)$$

Unacceptable Input

$$Y = 3X^2+15X-9$$

$$Y^{\wedge}4 = X$$

$$Y = LOG(X) \text{ for } X \leq 0$$

$$Y = 5E^{\wedge}X$$

$$Y = SINX$$

GRAPH INSTRUCTIONS (Page 3)

Polar Coordinates

- Step 1. Enter graph type - polar (option 2)
- Step 2. Decide if you want the default theta, that is do you want theta, Θ , to vary from 0 to 360 degrees (0 to 2π). On your first try, you will probably use the default theta; otherwise you will enter the values of theta you wish to use.
- Step 3. Decide if you want the axes to be automatically computed. On your first try, answer yes; later you may wish to specify them yourself.
- Step 4. Enter the maximum radius. This will be a number like 1, 2, 10, etc.
- Step 5. Enter the number of plot points. A larger number will give a more accurate graph, but will take longer to plot. Try 100 at first.
- Step 6. Enter the equation you want graphed. Remember to use BASIC notation. Use T for theta.
- Step 7. After the graph appears, decide whether to look at a portion of the graph using the options U(up), D(down), L(left), R(right), S(shrink), E(expand).
- Step 8. If you press the Space Bar, you may choose to plot this same graph on different axes or with a different number of points, or plot another graph on the same axes.

EXAMPLESAcceptable Input

$$R = \sin(2*T)$$

$$R = 3*\sin(T)$$

$$R = 1 + \cos(T)$$

$$R = 4$$

$$R = 1/T$$

$$R^2 = \cos(2*T)$$

Unacceptable Input

$$Y = \sin(2*X)$$

$$R = \sin T$$

$$R = \sin(2T)$$

$$R = \sin(T)$$

$$R = \sin(\text{THETA})$$

$$R = 3\sin(T)$$

$$R^2 = \cos(2T)$$

GRAPH INSTRUCTIONS (Page 4)

Parametric Equations

- Step 1. Enter the graph type - parametric (option 3)
- Step 2. Enter the lower limit of the parameter.
Enter the upper limit of the parameter.
The lower limit often is 0. The upper limit might be a multiple of π when using trigonometric functions or a time value, such as 60 seconds.
- Step 3. Enter the minimum horizontal x value.
Enter the maximum horizontal x value.
- Step 4. Decide if the y limits, the minimum and maximum vertical y values, should be automatically determined. On your first try, enter 'yes'. Later you may wish to enter the y values yourself.
- Step 5. Decide if you want equal spacing, that is, the scales on the x and y axes will be the same. Enter 'yes' on your first try.
- Step 6. Enter mean y value, the value of y to be at the center of the screen. You may wish to use zero in most cases.
- Step 7. Enter the number of plot points. A larger number gives a more accurate graph but takes longer to plot. Try 100 points.
- Step 8. Enter your equation for x, e.g. $X = T^2$
Enter your equation for y, e.g. $Y = T + 1$
Remember to use BASIC notation.
- Step 9. After the graph appears, decide whether to look at a portion of the graph using the options U(up), D(down), L(left), R(right), S(shrink), E(expand).
- Step 10. If you press the Space Bar, you may choose to plot this same graph on different axes or with a different number of points, or plot another graph on the same axes.

EXAMPLES

Acceptable Input

$$X = 3 * T, Y = 2 - T$$

$$X = \text{EXP}(T), Y = \text{EXP}(-T)$$

$$X = \text{SIN}(T), Y = \text{COS}(T)$$

$$X = T^2, Y = T + 1$$

Unacceptable Input

$$X = 3T, Y = 2 - T$$

$$X = E^T, Y = E^{-T}$$

$$X = \text{SIN}(Z), Y = \text{COS}(Z)$$

$$X = 2T, Y = T + 1$$

GRAPH WORKED EXAMPLES

1. Cartesian Coordinates

Problem: Graph $y = x^3 - 4x^2 + x + 6$

INPUT GRAPH TYPE

1-CARTESIAN

2-POLAR

3-PARAMETRIC

? 1

MINIMUM HORIZONTAL X VALUE? -10

MAXIMUM HORIZONTAL? 10

DO YOU WANT THE LIMITS COMPUTED? YES

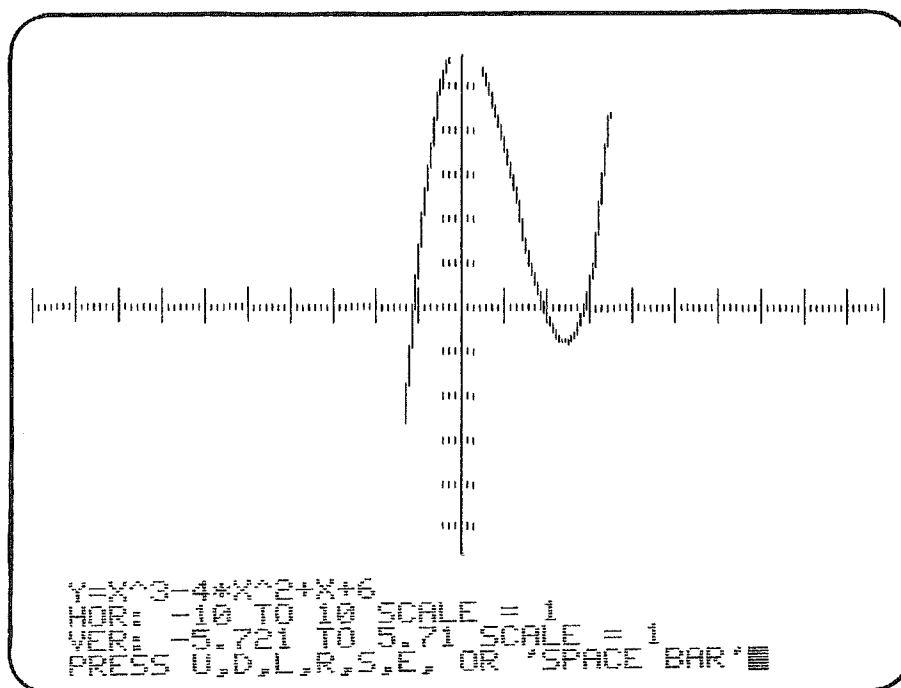
EQUAL SPACING? Y

MEAN Y VALUE? 0

NUMBER OF PLOT POINTS (> 49)? 100

INPUT YOUR EQUATION? $Y=X^3 - 4*X^2 + X + 6$

$Y=X^3-4*X^2+X+6$



GRAPH WORKED EXAMPLES

2. Polar Coordinates

Problem: Graph $r = \sin(2\theta)$.

INPUT GRAPH TYPE:

1 - CARTESIAN

2 - POLAR

3 - PARAMETRIC

? 2

DO YOU WANT THE DEFAULT THETA (0-360)? Y

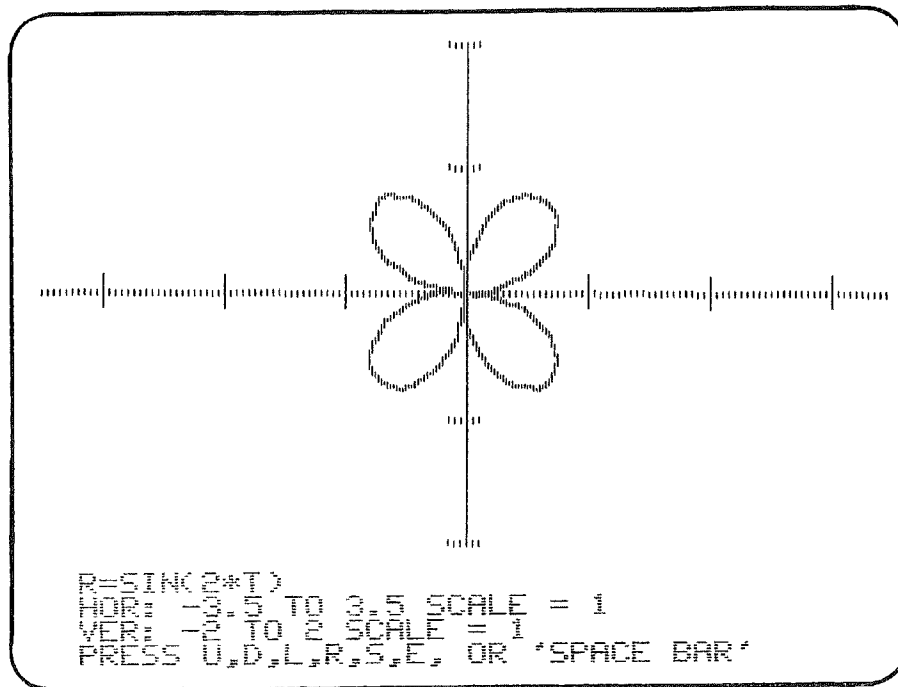
DO YOU WANT THE AXES COMPUTED? Y

WHAT IS THE MAXIMUM RADIUS? 2

NUMBER OF PLOT POINTS? 100

INPUT YOUR EQUATION? $R = \sin(2 * T)$

$R = \sin(2 * T)$



GRAPH WORKED EXAMPLES

3. Parametric Equations

Problem: Graph $x = \phi - \sin(\phi)$, $y = 1 - \cos(\phi)$

INPUT GRAPH TYPE

1-CARTESIAN

2-POLAR

3-PARAMETRIC

? 3

LOWER LIMIT OF PARAMETER? 0

UPPER LIMIT OF PARAMETER? 6.28

MINIMUM HORIZONTAL X VALUE? 0

MAXIMUM HORIZONTAL? 10

DO YOU WANT THE LIMITS COMPUTED? Y

EQUAL SPACING? Y

MEAN Y VALUE? 0

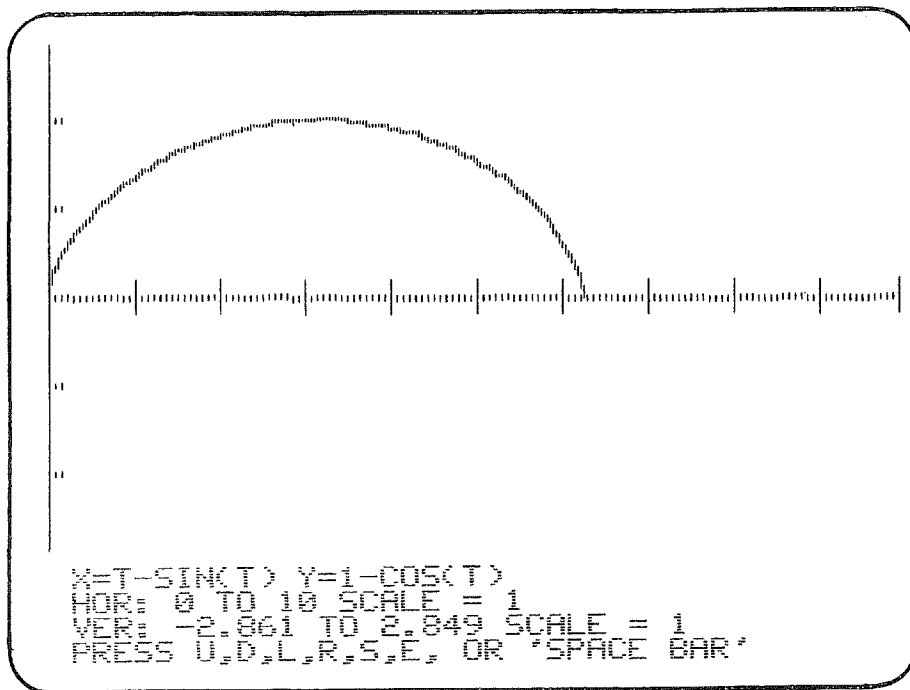
NUMBER OF PLOT POINTS (>49)? 100

INPUT YOUR EQUATION FOR X? $X = T - \sin(T)$

$X = T - \sin(T)$

INPUT YOUR EQUATION FOR Y? $Y = 1 - \cos(T)$

$X=T-\sin(T)$, $Y=1-\cos(T)$



CARTESIAN COORDINATES GRAPHING

Using GRAPH, plot each of the following curves. Record your results.

1.

$$y = x^2$$

$$y = 2x^2$$

$$y = -3x^2$$

$$y = \frac{1}{2}x^2$$

$$y = -\frac{1}{4}x^2$$

2.

$$y = x^2 - x - 6$$

$$y = \frac{x^2 - x - 6}{x}$$

$$y = \frac{x^2 - x - 6}{x - 3}$$

$$y = \frac{x^2 - x - 6}{x^2}$$

3.

$$y = \sin(x)$$

$$y = \cos(x)$$

$$y = \sin(x) + \cos(x)$$

$$y = 2 \sin(x)$$

$$y = \sin(2x)$$

POLAR COORDINATE GRAPHING

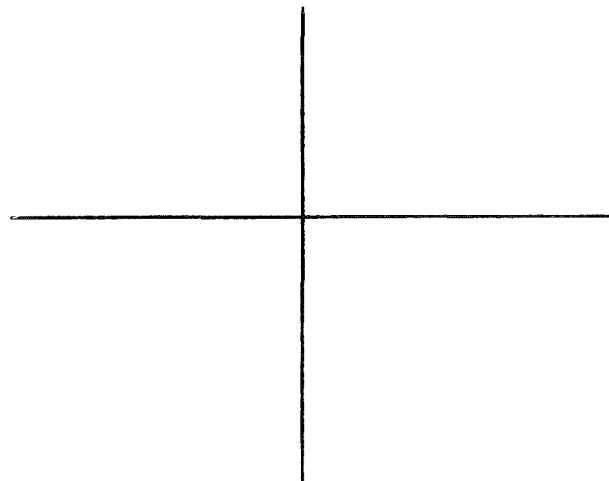
Using GRAPH, plot each of the following curves. Record your results on this page or on graph paper. (T means theta, Θ).

1. Cardioids

$$R = 1 + \cos(T)$$

$$R = 1 - \sin(T)$$

$$R = 3 * \cos(T)$$



Write another equation for a cardoid and then graph it.

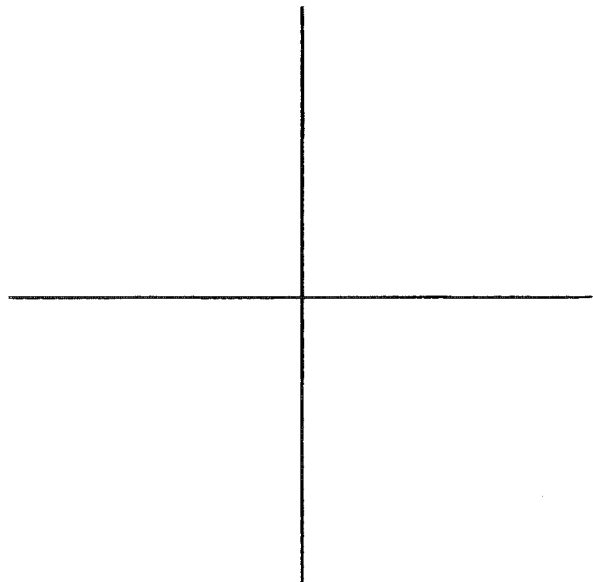
2. Roses

$$R = \sin(2*T)$$

$$R = \sin(3*T)$$

$$R = \cos(5*T)$$

$$R = \cos(4*T)$$



Write another equation for a rose. Predict how many 'leaves' it will have, then graph it.

GRAPHING PARAMETRIC EQUATIONS

Using GRAPH, plot the following equations, record your results on graph paper.

1. A 5 cuspid epicycloid

$$x = 6 \cos(t) - \cos(6t)$$

$$y = 6 \sin(t) - \sin(6t)$$

$$X = 6 \cdot \cos(T) - \cos(6 \cdot T)$$

$$Y = 6 \cdot \sin(T) - \sin(6 \cdot T)$$

2. $x = \cos(t)$, $0 \leq t \leq 2$
 $y = \sin(t)$

$$X = \cos(T)$$

$$Y = \sin(T)$$

$$0 \leq T \leq 2$$

How does this graph compare to the graph of $x^2 + y^2 = 1$?

3. $x = t + 1$, $0 \leq t \leq \infty$

$$y = t^2 + 4$$

$$X = T + 1 \quad 0 \leq T \leq \infty$$

$$Y = T^2 + 4$$

GRAPH

SAMPLE RUNS

GRAPH
IS A PROGRAM DESIGNED TO PLOT RELATIONS
GIVEN IN THE FOLLOWING FORMS:

CARTESIAN

$Y = F(X)$ -OR- $Y^2 = \dots$

POLAR

$R = F(T)$ -OR- $R^2 = \dots$

PARAMETRIC

$X = F(T)$ AND $Y = G(T)$

DO YOU WANT INSTRUCTIONS?

The various types of graphs available are listed. The user may request instruction.

EXAMPLES OF SCREEN OUTPUT

INPUT GRAPH TYPE:
1 - CARTESIAN
2 - POLAR
3 - PARAMETRIC
??

The user selects the type of graph to be used.

GRAPH

SAMPLE RUNS

DO YOU WANT THE DEFAULT THETA (0-360)?Y

DO YOU WANT THE AXES COMPUTED?Y

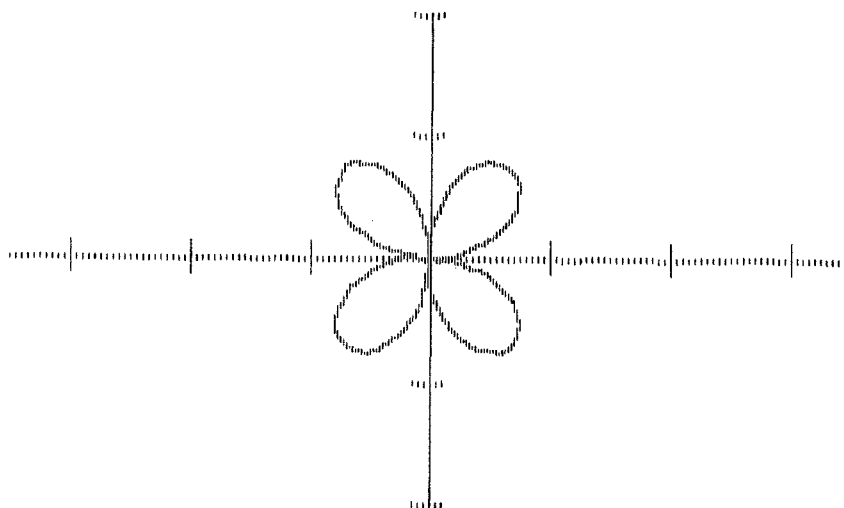
WHAT IS THE MAXIMUM RADIUS?2

NUMBER OF PLOT POINTS (> 49)?50

INPUT YOUR EQUATION ?R=SIN(2*T)

The user enters the necessary data when requested.

EXAMPLES OF SCREEN OUTPUT



R=SIN(2*T)
HOR: -3.5 TO 3.5 SCALE = 1
VER: -2 TO 2 SCALE = 1
PRESS U,D,L,R,S,E, OR 'SPACE BAR'

The graph is plotted on the screen. The user may then view portions of the graph or plot another graph on the same axes.

F U N C T I O N L I M I T S

LIMITS OF A FUNCTION OF TWO VARIABLES

Specific Topic: Calculus of several variables

 Type: Problem Solving

Reading Level: 9-10 (Dale-Chall)

 Grade Level: Higher Education

DESCRIPTION...

This program investigates the limit of a function of two variables, $z=f(x,y)$ at a point specified by the student along the path determined by the student. It can also be used with functions of one variable by considering $z=f(x)$ with $y=0$ as the path of approach.

OBJECTIVES...

1. to determine if a limit exists for a function of two variables.
2. to determine the limit of a function of two variables, if it exists.
3. to study the continuity of functions of two variables.
4. to investigate a discontinuity by considering several paths of approach.

FUNCTION LIMITS

BACKGROUND INFORMATION...

The Mathematical Method

If the limit of a function exists, it has the same value regardless of the path of approach. If different values are found, then the limit fails to exist. That is,

$$\text{if } \lim_{x \rightarrow a} f(x, g(x)) = \lim_{x \rightarrow b} f(g(y), y)$$

for all paths $y=g(x)$ and all paths $x=g(y)$,

then $\lim_{(x,y) \rightarrow (a,b)} f(x,y)$ exists and has the same value.

The Computer Method

The function is entered first, then the path of approach and the point (a,b) are entered.

The function and the path must be described using BASIC notation for operations and elementary functions. (See handout #1 - Mathematical Expressions in BASIC.)

If the function is $f(x,y)$ and the path is $g(x)$, the search begins within .9 units of a the x-coordinate of the point (a,b), and continues with smaller values of Δx , until the successive function values of f differ by less than .000001. Then the limit on that path is indicated as the final function value.

If during the search the value of Δx is reduced to .000000001, then the search is stopped and the program concludes that no limit can be obtained. In such an instance either the function has no limit or the function values are changing too rapidly for this program to consider.

This program can also be used to compute the limit of a function of one variable. For example, to find

$$\lim_{x \rightarrow 0} \sin(x)/x,$$

use $F = \text{SIN}(X)/X$ as the function
 $Y = 0$ as the path
 $0,0$ as the point
either 1 or 2 as the approach option

The limit computed is .999999333.

FUNCTION LIMITS

USE IN AN INSTRUCTIONAL SETTING...

Preparation

Students should be familiar with the concept of limit. They should have a copy of handout #7 - FUNCTION LIMITS Instructions available when using the program. They should see a demonstration of the program in class or have a copy of handout #8 - FUNCTION LIMITS Worked Examples.

Using the Program

The students may be given handout #9 - FUNCTION LIMITS Applications or may use problems from a textbook.

FUNCTION LIMITS can be used in the study of limits of functions of several variables and the study of continuity. Several paths can be quickly considered to determine a limit or lack of it.

$$\text{e.g., } f(x,y) = \frac{xy}{x^2+y^2} \quad \text{at } (a,b) = (0,0) \text{ along the paths } y = 0, x = 0, \\ y = 2x, \text{ etc.}$$

$$f(x,y) = \frac{x^2y}{x^4+y^2} \quad \text{at } (a,b) = (0,0) \text{ along the paths } y = 0, x = 0, \\ y = 2x, y = x^2, \text{ etc.}$$

When a limit does not exist, the nature of the discontinuity can be analyzed by considering several paths.

FUNCTION LIMITS INSTRUCTIONS

FUNCTION LIMITS will search for a limit of any function of two variables, x and y.

Step 1. Enter the function you wish to investigate

e.g. $f(x,y) = xy/(x^2 + y^2)$ type $X*Y/(X^2 + Y^2)$

Step 2. Enter the path of approach that you choose by typing in either

$y = (\text{a function of } x)$ or $x = (\text{a function of } y)$

e.g. $y = 2x$ type $Y = 2*X$

$x = y + 1$ type $X = Y + 1$

Step 3. Enter the point you wish to consider. Remember that this point must lie on the path of approach.

e.g. $(0,0)$ on $y=2x$, type 0,0

$(1,0)$ on $x=y+1$, type 1,0

Step 4. Choose the direction of approach.

- 1) approach the point from the right (or above)
- 2) approach the point from the left (or below)

type 1 or 2

NOTE: This program will also search for the limit of a function of one variable, x.

Step 1. Consider the function z to be $f(x)$

e.g. $z = f(x) = \frac{x}{x-1}$

Step 2. Use the function $y=0$ as the path of approach.

Step 3. Enter the point $(a,0)$, using the appropriate value of a depending on the function z.

Step 4. Choose either direction of approach, as described above.

FUNCTION LIMITS WORKED EXAMPLES

Function of Two Variables

ENTER FUNCTION OF X AND Y
 $F = X*Y / (X*X + Y*Y)$
 ENTER THE PATH FUNCTION AS
 $Y = \dots \text{FN OF } X \quad \text{OR} \quad X = \dots \text{FN OF } Y.$
 $Y = 2*X$
 ENTER POINT (A,B) ? 0,0
 DO YOU WANT TO
 1. APPROACH (0,0) FROM THE RIGHT (OR ABOVE)
 2. APPROACH (0,0) FROM THE LEFT (OR BELOW)
 ENTER 1 OR 2 ? 1
 COMPUTING
 THE LIMIT OF YOUR FUNCTION $(X*Y) / (X*X + Y*Y)$
 ALONG THE PATH $Y = 2*X$ IS .4

DO YOU WISH TO
 1) USE THE SAME FUNCTION WITH A DIFFERENT PATH
 2) RUN THE PROGRAM AGAIN USING A DIFFERENT FUNCTION
 3) RETURN TO MENU
 ENTER 1, 2, OR 3 ? 3

Function of One Variable

ENTER FUNCTION OF X AND Y
 $F = (X^2 - X - 6) / (X - 3)$
 ENTER THE PATH FUNCTION
 $Y = 0$
 ENTER POINT (A,B) ? 3,0
 DO YOU WANT TO
 1) APPROACH (3,0) FROM THE RIGHT (OR ABOVE)
 2) APPROACH (3,0) FROM THE LEFT (OR BELOW)
 ENTER 1 OR 2 ? 1
 COMPUTING
 THE LIMIT OF YOUR FUNCTION $(X^2 - X - 6) / (X - 3)$
 ALONG THE PATH $Y = 0$ IS 5.

FUNCTION LIMITS APPLICATIONS

1. Use FUNCTION LIMITS to show that

$$f(x,y) = \frac{2xy}{x^2 + y^2}, \quad (x,y) \neq (0,0) \text{ and } f(0,0) = 0$$

is not continuous at $(0,0)$.

Hint: Show that $\lim_{(x,y) \rightarrow (0,0)} f(x,y)$ does not exist.

2. Find $\lim_{x \rightarrow 4} \frac{\sqrt{x} - 2}{x - 4}$

Hint: Let $F = (\text{SQR}(X)-2)/(X-4)$

FUNCTION LIMIT ANSWERS

1. Using path $Y=X$, and approaching from the right, the limit is .999999999.
Using path $Y=X$, and approaching from the left, the limit is .999999999.
Using path $Y=0$, and approaching from the right, the limit is 0.
2. Use path $Y=0$, point (4.0),
if approach from right, the limit is .249989356;
if approach from left, the limit is .250006403.

FUNCTION LIMITS

SAMPLE RUNS

```
ENTER FUNCTION OF X AND Y.  
  F = X*Y/(X^2+Y^2)  
ENTER THE PATH FUNCTION AS  
Y = ...FN OF X  OR  X = ...FN OF Y.  
  Y=2*X  
ENTER POINT (A,B)? 0,0  
DO YOU WANT TO  
  1) APPROACH (0,0) FROM THE  
    RIGHT (OR ABOVE)  
  2) APPROACH (0,0) FROM THE  
    LEFT (OR BELOW)  
ENTER 1 OR 2?
```

The user enters the function to be examined, the path chosen, the point at which the limit is to be studied, and the direction of approach.

EXAMPLES OF SCREEN OUTPUT

```
THE LIMIT OF YOUR FUNCTION X*Y/(X^2+Y^2)  
ALONG THE PATH Y=2*X IS .4
```

The limit is computed and printed on the screen. The user may then choose to examine the same function along a different path or use a different function.

```
PRESS  TO CONTINUE
```

ROOTS OF POLYNOMIAL EQUATIONS

Specific Topic: College algebra, analytic geometry, calculus, complex analysis

Type: Problem Solving

Reading Level: 9-10 (Dale-Chall)

Grade Level: 10 - Higher Education

DESCRIPTION...

POLYROOT will locate all real and complex roots of any real polynomial of degree 32 or less, using Newton-Raphson iterative techniques.

OBJECTIVES...

1. to determine the roots of polynomial equations.
2. to find the roots of characteristic polynomials in linear algebra.
3. to find the roots of unity in complex analysis.
4. to study iterative techniques in numerical analysis.
5. to demonstrate a use of Cauchy Riemann equations.

BACKGROUND INFORMATION...

The Mathematical Method

Given a polynomial $f(z) = a_n z^n + \dots + a_1 z + a_0$, let $U(x,y)$ be the real terms and $V(x,y)$ the imaginary terms. Let $z = x + iy$ be an initial approximation of a root. In $z^n = (x+iy)^n$ let x_n be the real terms and y_n the imaginary terms. Then for $n = 0$: $x_0 = 1, y_0 = 0$

$$\begin{aligned} \text{for } n > 0: x_n &= x x_{n-1} - y y_{n-1} \\ y_n &= x y_{n-1} + y x_{n-1} \end{aligned}$$

$$U = \sum_{k=0}^N a_k x_k \quad \text{and} \quad V = \sum_{k=1}^N a_k y_k$$

$$\text{Now } U_x = \sum_{k=1}^N k x_{k-1} a_k \quad \text{and} \quad V_y = - \sum_{k=1}^N k y_{k-1} a_k.$$

From Newton-Raphson $z_{j+1} = z_j - f(z_j)/f'(z_j)$ for $f'(z_j) \neq 0$,

$$\begin{aligned} \frac{f(z)}{f'(z)} &= \frac{U + iV}{U_x + iV_x} \\ &= \frac{U + iV}{U_x - iU_y} \quad \text{by Cauchy Riemann equations,} \\ &= \Delta x + i\Delta y. \end{aligned}$$

$$\text{Expanding we find } \Delta x = \frac{-VU_y + UU_x}{U_x^2 + U_y^2}, \quad \Delta y = \frac{UU_y + VU_x}{U_x^2 + U_y^2}$$

and the next approximation is

$$x_{j+1} = x_j - \Delta x, \quad y_{j+1} = y_j - \Delta y.$$

BACKGROUND INFORMATION (Continued)...The Computer Method

The program will select an initial value near zero and perform up to 500 iterations of the method. If a root is not found, the program obtains another initial root and again performs up to 500 iterations. This continues for 5 such initial values or until a root is found.

When a root is found, a reduced polynomial is obtained for which the next root is found. The final iterations for any root are performed using the original polynomial, not the reduced polynomial.

With some high degree polynomials, the program may end before completion, because numbers computed in the model are too large for Applesoft BASIC.

Polynomials of high degree (near 32) require a long time for computation. For example, $Y = X^{32}-1$ can be used but the roots will be found only after several minutes of computation time.

USE IN AN INSTRUCTIONAL SETTING...

Preparation

The students should understand the concept of a root of a polynomial and have experience in working with polynomials.

A classroom demonstration of this program and handouts #10 - POLYROOT Instructions and #11 - POLYROOT Worked Examples would be helpful to first-time users.

Using the Program

The program can be used in any course where a student must find roots of polynomials that are not simple or where application of an algorithm is time consuming.

Handout #12 - POLYROOT Applications presents several sample problems.

Other examples include:

1. Linear algebra
Roots of characteristic polynomials can be found.
2. Complex analysis
Roots of unity can be found. The algorithm is based on use of the Cauchy Riemann equations.
3. Numerical analysis
Iterative techniques and solutions of nonlinear equations can be studied.

The program GRAPH can be used with this program to plot the function whose roots are being computed. Graphing the function can help the student understand the meaning of a root of the function.

POLYROOT INSTRUCTIONS

POLYROOT will locate all real and complex roots of any real polynomial of degree 32 or less, using Newton-Raphson iterative techniques.

Step 1. Enter the degree of the polynomial (≤ 32).

e.g. 4

Step 2. Enter the coefficients (including zeros). The program asks you to enter the coefficients in order, beginning with the highest degree term. Be sure to include zero as the coefficient when a term is not present.

e.g. $P(X) = 1X^4 - 2X^3 + 5X^2 + 0X + .5X^0$

Step 3. Read the roots from the table shown on the screen.

e.g.	<u>Real Part</u>	<u>Complex Part</u>
	4	-3

This means that the root is $4 - 3i$.

POLYROOT WORKED EXAMPLE

Problem: Find the roots of $x^3 + x^2 - 2 = 0$

ENTER DEGREE OF POLYNOMIAL (32 MAX) ? 3

ENTER COEFFICIENTS (INCLUDING ZEROS)

$P(X) = ? X^3$

$P(X) = 1 X^3 + ? X^2$

$P(X) = 1 X^3 + 1 X^2 + ? X^1$

$P(X) = 1 X^3 + 1 X^2 + 0 X^1 + ? X^0$

$P(X) = 1 X^3 + 1 X^2 + 0 X^1 - 2 X^0$

ROOTS OF THE POLYNOMIAL

REAL PART

COMPLEX PART

-1.00000001

1

-1.00000001

-1

1.00000001

0

INTERPRETATION OF RESULTS

The roots are

$-1 + i,$

$-1 - i,$

and 1.

POLYROOT APPLICATIONS

1. Use POLYROOT to find the roots of $x^2 - x - 6 = 0$

Real Part

Complex Part

2. Use POLYROOT to find the cube roots of 5. Hint: If $x = \sqrt[3]{5}$, then $x^3=5$ or $x^3-5=0$. Let $f(x) = x^3-5$ and find the roots of f .

Real Part

Complex Part

3. If the number of mosquitos, $M(x)$, in millions in a given area depends on the June rainfall, x , measured in inches, and is approximated by

$$M(x) = -x^3 + 14x^2 - 32x + 50.$$

Find the amounts of rainfall that will produce the maximum and the minimum number of mosquitos.

Hint: Find the roots of the derivative of $M(x)$, $M'(x)$.

POLYROOT ANSWERS

1.	<u>Real Part</u>	<u>Complex Part</u>
	-2.0	0
	3.0	0
2.	1.70997597	0
	-.854987973	-1.48088261
	-.854987973	1.48088261
3.	$M'(x) = -3x^2 + 28x - 32$	
	1.333333333	0
	8	0

Maximum number of mosquitos when rainfall is 8 inches; minimum when rainfall is 1-1/3 inch.

P O L Y R O O T

SAMPLE RUNS

ENTER COEFFICIENTS (INCLUDING ZEROS)
 $P(X) = 1 X^2 + 0 X^1 - 2 X^0$

The user enters the degree of the polynomial and the coefficients of all terms (including zeros).




EXAMPLES OF SCREEN OUTPUT

ENTER COEFFICIENTS (INCLUDING ZEROS)
 $P(X) = 1 X^2 + 0 X^1 - 2 X^0$

ROOTS OF THE POLYNOMIAL:

REAL PART	COMPLEX PART
-1.41421356	0
1.41421356	0

The real and complex roots are printed out.

PRESS  TO CONTINUE

NEWTON'S METHOD FOR APPROXIMATING ROOTS OF EQUATIONS

Specific Topic: Calculus, numerical analysis, analytic geometry

Type: Algorithm Development, Problem Solving

Reading Level: 9-10 (Dale-Chall)

Grade Level: Higher Education

DESCRIPTION...

The program will calculate a sequence of approximations to a root of an equation of the form $f(x) = 0$, where f is a differentiable function. The method used is Newton's method for finding successive approximations to the root. The student determines the number of approximations to be computed.

OBJECTIVES...

1. to find the root of a nonlinear function by considering successive approximations using Newton's method.
2. to observe an application of derivatives.

BACKGROUND INFORMATION...

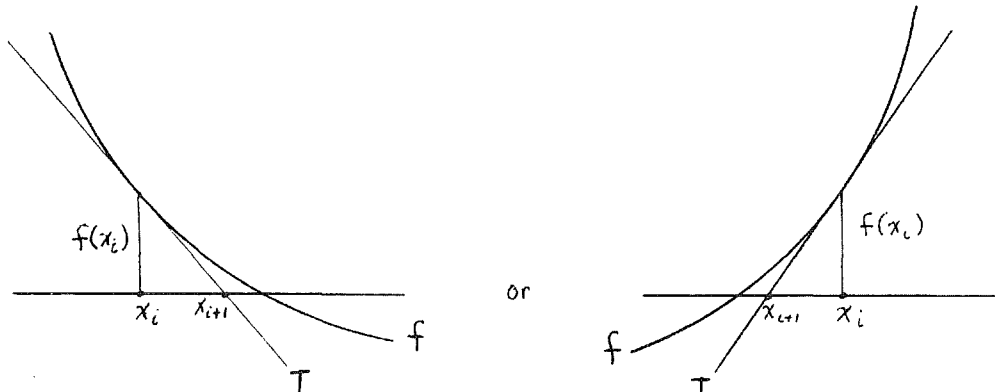
The Mathematical Method

Given the equation $f(x) = 0$ with $f'(x)$ as the derivative of $f(x)$, Newton's method of successive approximations to a root of the equation is

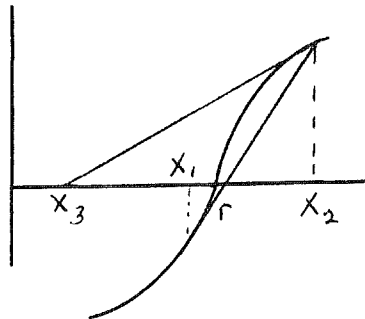
$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \quad \text{for the (i+1)st approximation}$$

This is based on the fact that the tangent line to the curve at the point $(x_i, f(x_i))$ crosses the x-axis at x_{i+1} .

$$\text{Hence } f'(x_i) = f(x_i)/(x_i - x_{i+1}) \quad \text{or} \quad f'(x_i) = -f(x_i)/(x_{i+1} - x_i).$$



In some cases, Newton's method may not converge. For example, the graph below shows a function such as $f(x) = \sqrt{x-r}$ for $x \geq r$ and $-\sqrt{r-x}$ for $x \leq r$.



No further iterations will improve the approximation.

BACKGROUND INFORMATION (Continued)...The Computer Method

The user enters any differentiable function and its derivative. These functions must be in BASIC notation. (See handout #1 - Mathematical Expressions in BASIC.)

The computer program does not check that the derivative function is correct. The instructor may wish to provide the student with the derivative if the function is difficult to differentiate.

The initial approximation must be entered by the user. The computer calculates and prints out both the current approximation and the value of the function at that approximation. After any approximation is printed, the user may request a further approximation or may end the program.

If the derivative is found to be zero, the program will end, since $f(x)/f'(x)$ is not defined.

USE IN AN INSTRUCTIONAL SETTING...

Preparation

The student should understand the algorithm used in Newton's method. The function used on handout #14 - NEWTON 1 Worked Example is simple enough to be calculated by hand to two decimal places. Graphic representations of Newton's method, such as those found on handout #13 - NEWTON 1 Instructions, are useful.

Students may find it helpful to graph the function using the program GRAPH.

Using the Program

Newton's method is a simple application of derivatives and provides students with a handy tool for obtaining approximate roots of equations that are otherwise difficult to find. The program allows the student to see the individual iterations of Newton's method and obtain any desired accuracy.

A variety of initial approximations can be used to explore the effect on the number of iterations required to reach a given accuracy.

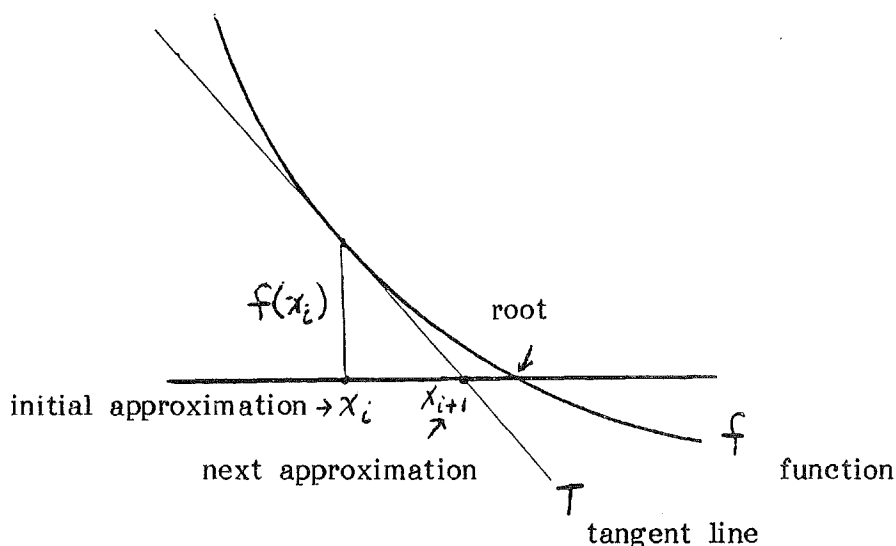
It should be pointed out that Newton's method produces only one root of the equation, the one closest to the initial approximation. If the student wishes to find all the roots of a polynomial function, the program POLYROOT will provide the information more directly.

If after several iterations it appears that there is no convergence of the approximations, the student should try the program NEWTON 2 which will allow quick analysis using as many iterations as desired.

NEWTON 1 INSTRUCTIONS

NEWTON 1 uses Newton's method to find an approximate root to any differentiable function.

- Step 1. Enter a function of x .
- Step 2. Enter the function's derivative. Be sure you have calculated the derivative correctly. (Check with the instructor if necessary.)
- Step 3. Enter an initial approximation to the root of the function.
- An initial approximation may be found by graphing (use the program GRAPH) or by finding values a and b such that $f(a)$ and $f(b)$ have opposite signs and using $(a + b)/2$ as an initial approximation.
- Step 4. Request further approximations of the root until the function value equals or approaches zero.
- Step 5. If desired, run the program again, using a different initial approximation.



NEWTON 1 WORKED EXAMPLE

Problem: Find an approximate root of $y = x^2 - 5$.

ENTER THE FUNCTION IN THE FORM

F = ... FUNCTION OF X

F = $X^2 - 5$

ENTER THE DERIVATIVE OF F IN THE FORM

D = ... DERIVATIVE FUNCTION

D = $2 * X$

INITIAL ROOT APPROXIMATION ? 2

F = $X^2 - 5$

D = $2 * X$

ROOT APPROXIMATION	FUNCTION VALUE
2	-1

WOULD YOU LIKE ANOTHER APPROXIMATION ? (YES OR NO) Y

2.25

.062500009

2.2361111

1.92906708E-04

2.23606798

0

2.23606798

0

DO YOU WANT TO

1) TRY ANOTHER EQUATION

2) RETURN TO THE MENU

ENTER 1 OR 2 ?

INTERPRETATION OF RESULTS

The root of $f(x)=x^2-5$ nearest to the initial approximation of 2, is approximately 2.236 (to four significant figures). That is, $f(2.236)$ is approximately equal to zero, $f(2.236) \approx 0$. (There is of course, another root of $f(x) = x^2-5$, -2.236.) Notice that the root of this equation is an approximation of the square root of five.

NEWTON 1 APPLICATIONS

1. Newton's method may be used to calculate cube roots, fourth roots, etc.

For example, find the cube root of 2 by finding the root of the equation

$$x = \sqrt[3]{2} \quad \text{or} \quad x^3 = 2$$

Let $f(x) = x^3 - 2$ and use NEWTON 1

Approximations

Function Value

2. Determine a root of $f(x) = x^3 + x - 1$ to four significant figures, using NEWTON 1

For what value of x does $f(x) = 0$? _____.

3. Find the root of $f(x) = 2 - e^x$ to four significant figures.

4. Use NEWTON 1 to examine the function $f(x) = e^{1/x}$. Hint: $f'(x) = \frac{-e^{1/x}}{x^2}$

Does it have a real root? What value does f approach for very large values of x ? What is the limit $e^{1/x}$?
 $x \rightarrow \infty$

NEWTON 1 ANSWERS

ANSWER KEY

1.	<u>APPROXIMATION</u>	<u>FUNCTION VALUE</u>
	1	-1
	1.33333333	.370370375
	1.26388889	.0189552223
	1.25993349	5.92591241E-05
	1.25992105	9.31322575E-10
2.	0	-1
	1	1
	.75	.171875
	.686046512	8.94103711E-03
	.682339582	2.82307156E-05
	.682327804	0
3.	.6931	
4.	1	2.718 . . .
	2	1.648 . . .
	42	1.024
	1806	1.0005

Notice that the approximation or x value is getting much larger while the function value approaches 1. There is no real root. The limit as $x \rightarrow \infty$ is 1.

NEWTON 1

SAMPLE RUNS

ENTER THE FUNCTION IN THE FORM

F = ...FUNCTION OF X

F = X^2-5

ENTER THE DERIVATIVE OF F IN THE FORM

D = ...DERIVATIVE FUNCTION

D = 2*X

ROOT APPROXIMATION?

The user enters the function and its derivative. Note that the program does not check for incorrect derivatives.

EXAMPLES OF SCREEN OUTPUT

F=X^2-5

D=2*X

ROOT
APPROXIMATION

FUNCTION
VALUE

2

-1

2.25

.0625000009

2.236111111

1.92906708E-04

2.23606798

0

2.23606798

0

WOULD YOU LIKE ANOTHER

APPROXIMATION (YES OR NO)?

The user may continue requesting further approximations until the function value is sufficiently close to zero.

NEWTON'S METHOD FOR APPROXIMATING ROOTS OF EQUATIONS_____

Specific Topic: Calculus, numerical analysis, analytic geometry

Type: Problem Solving

Reading Level: 9-10 (Dale-Chall)

Grade Level: Higher Education

DESCRIPTION...

The program will calculate several iterations in Newton's method of finding successive approximations to a root of an equation of the form $f(x) = 0$, where f is a differentiable function. Only the final value of the approximation procedure is printed.

OBJECTIVES...

1. to find the root of a differentiable function using Newton's method.
2. to observe an application of derivatives.

BACKGROUND INFORMATION...

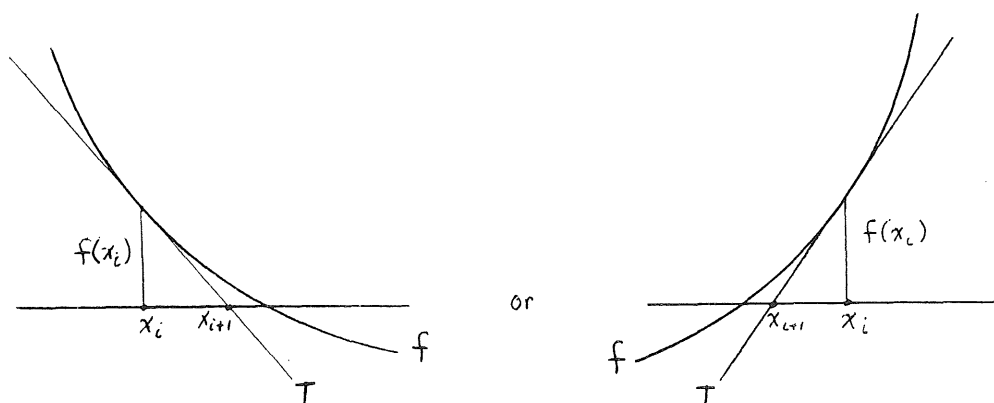
The Mathematical Method

Given the equation $f(x) = 0$ with $f'(x)$ as the derivative of $f(x)$, Newton's method of successive approximations to a root of the equation is

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \quad \text{for the } (i+1)\text{st approximation}$$

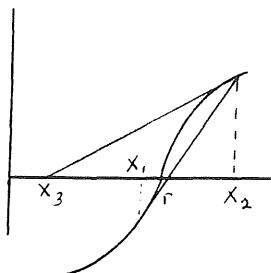
This is based on the fact that the tangent line to the curve at the point $(x_i, f(x_i))$ crosses the x-axis at x_{i+1} .

Hence $f'(x_i) = f(x_i)/(x_i - x_{i+1})$ or $f'(x_i) = -f(x_i)/(x_{i+1} - x_i)$.



In some cases, Newton's method may not converge. For example, the graph below shows a function such as $f(x) = \sqrt{x-r}$ for $x \geq r$ and $f(x) = -\sqrt{r-x}$ for $x \leq r$.

No further iterations will improve the approximation.



BACKGROUND INFORMATION (Continued)...

The Computer Method

The user enters the function $f(x)$ and its derivative $f'(x)$.

The user also enters the maximum number of iterations the program should consider.

The program searches until $f(x_i)/f'(x_i) \leq .000001$. Then the final approximation and the value of the function at that point are printed as the final result. If $f(x_i)/f'(x_i)$ is not less than .000001 after that maximum number of iterations, the program terminates after printing the last approximation and the function value at that point.

If the derivative is found to be zero, the program terminates since $f(x)/f'(x)$ is not defined for zero.

Notice that NEWTON 2 prints only the final value of the approximation procedure, while NEWTON 1 prints each successive approximation.

USE IN AN INSTRUCTIONAL SETTING...

Preparation

The student will benefit from first using NEWTON 1, which lists each successive approximation. The student should be familiar with Newton's method.

A classroom demonstration or copies of handouts #16 - NEWTON 2 Instructions and #17 - NEWTON 2 Worked Example may be necessary for a first-time user of this program.

Students may find it helpful to graph the function using the program GRAPH.

Using the Program

In numerical analysis, this program can be used to compare Newton's method with other methods of solving nonlinear equations.

It should be pointed out that Newton's method produces only one root of the equation, the one closest to the initial approximation. If the student wishes to find all the roots of a polynomial function, the program POLYROOT will provide the information more directly.

In calculus, it can be used as an application of derivatives. It can also be used to compare Newton's method with other approximation methods, such as differentials of Taylor polynomials.

NEWTON 2 may also be used in any course where the student needs to solve nonlinear equations. For example, to find the intersection of

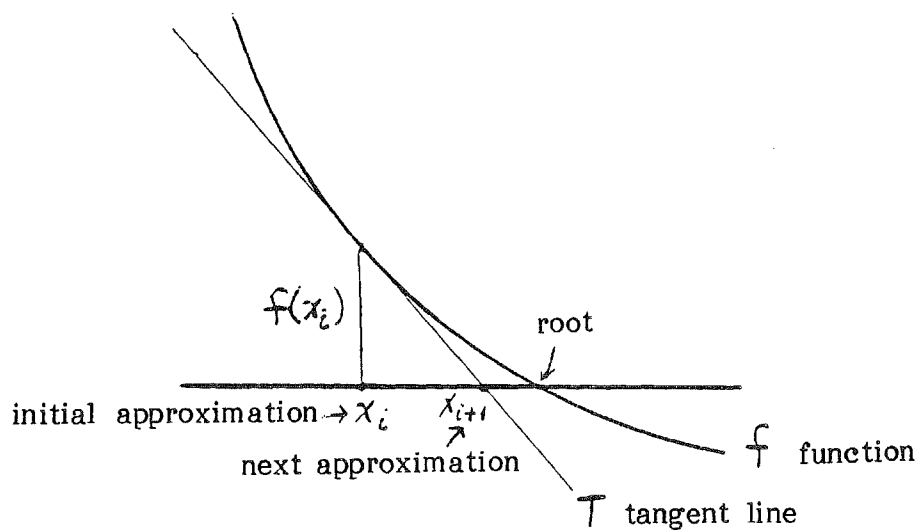
$x^2 + y^2 = 4$ and $y = x^3$. The equation to be solved is $x^2 + x^6 - 4 = 0$.

Handout #18 - NEWTON 2 Applications provides some sample problems.

NEWTON 2 INSTRUCTIONS

NEWTON 2 uses Newton's method to find approximate roots to any differentiable function.

- Step 1. Enter a function.
- Step 2. Enter its derivative. Be certain to calculate the derivative correctly.
- Step 3. Enter an initial approximation to the root of the function.
- Step 4. Enter the maximum number of iterations to be performed.



NEWTON 2 WORKED EXAMPLE

Problem: Find an approximate root to $f(x) = x^4 + x - 3$.

ENTER THE FUNCTION IN THE FORM

F = ... FUNCTION OF X

$$F = X^4 + X - 3$$

ENTER THE DERIVATIVE OF F IN THE FORM

D = ... DERIVATIVE FUNCTION

$$D = 4 * X^3 + 1$$

INITIAL APPROXIMATION ? 0

MAXIMUM NUMBER OF ITERATIONS ? 10

$$F = X^4 + X - 3$$

$$D = 4 * X^3 + 1$$

ROOT
APPROXIMATION

1.16403515

FUNCTION
VALUE

1.86264515E-09

AFTER 8 ITERATIONS

INTERPRETATION OF RESULTS

The root of the function $f(x) = x^4 + x - 3$ nearest to the initial approximation of 0 is 1.164 (to four significant figures). That is, $f(1.164)$ is approximately equal to zero, $f(1.164) \doteq 0$. (There are, of course, other roots of $f(x) = x^4 + x - 3$.)

NEWTON 2 APPLICATIONS

1. Find the intersection of the equations

$$x^2 + y^2 = 4 \quad \text{and} \quad y = x^3$$

Hint: Substitute the expression for y from the second equation into the first equation. Then you will have one equation in x . Use NEWTON 2 to find its roots.

Then graph the two equations using the program GRAPH, and read off the point of intersection.

2. Use NEWTON 2 to show that $f(x) = \sin(x)$ crosses the x -axis at $-\pi$, 0 , π , 2π , etc. Then plot this function on the interval $(-2\pi, +2\pi)$ using the program GRAPH.

NEWTON 2 ANSWERS

1. $x^2 + (x^3)^2 = 4$

$$f(x) = x^6 + x^2 - 4$$

initial approximation 1
root approximation 1.1742218
function value 8.45640898E-07
after 4 iterations.

2. $F = \sin(X), \quad D = \cos(X)$

initial approximation -3
root approximation -3.14159265
function value -7.3145904E-10

initial approximation 0
root approximation 0
function value 0

initial approximation 3
root approximation 3.14159265
function value 7.3145904E-10

initial approximation 6
root approximation 6.28318515
function value -1.57995152E-07

NEWTON 2

SAMPLE RUNS

ENTER THE FUNCTION IN THE FORM

F = ...FUNCTION OF X

F = $X^4 + X - 3$

ENTER THE DERIVATIVE OF F IN THE FORM

D = ...DERIVATIVE FUNCTION

D = $4 * X^3 + 1$

INITIAL ROOT APPROXIMATION?

The user enters the function, its derivative, the initial root approximation and the maximum number of iterations.

EXAMPLES OF SCREEN OUTPUT

F= $X^4 + X - 3$

D= $4 * X^3 + 1$

ROOT
APPROXIMATION

FUNCTION
VALUE

1.16403514

1.86264515E-09

AFTER 8 ITERATION(S).

The root approximation and the value of the function at that root are printed. The number of iterations needed is also given.

I N T E G R A T I O N

NUMERICAL INTEGRATION - ONE VARIABLE

Specific Topic:	Calculus
Type:	Algorithm Development and Problem Solving
Reading Level:	9-10 (Dale-Chall)
Grade Level:	12 - Higher Education

DESCRIPTION...

This program calculates numerical approximations of a definite integral. The student enters the integrand and selects from several methods using Riemann sums, the trapezoidal rule, and the parabolic rule.

OBJECTIVES...

1. to obtain an approximation to a definite integral.
2. to study Reimann sums.
3. to study the trapezoidal rule.
4. to study the parabolic rule.
5. to use integration to determine arc length.
6. to use integration in probability theory.

INTEGRATION

BACKGROUND INFORMATION...

The Mathematical Method

Definite integrals of one variable can be numerically approximated by several methods. Those usually seen in a first year calculus course are Riemann sums, the trapezoidal rule, and the parabolic rule.

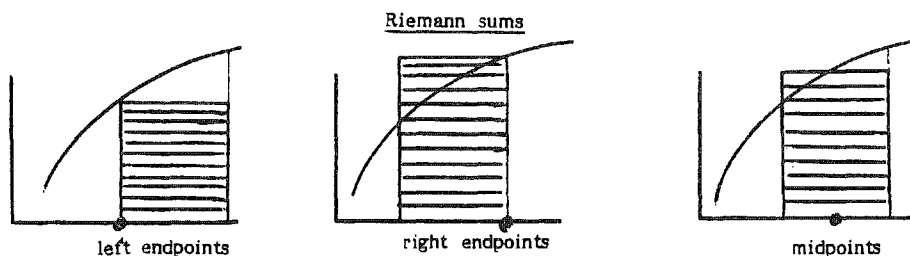
The integral of a continuous function $\int_a^b f(x) dx$ can be

approximated by the Riemann sum $\sum_{i=1}^n f(z_i) \Delta x$, where the

subintervals are all of equal length, $\Delta x = \frac{b-a}{n}$, and the points z_i are chosen in a manner similar for all subintervals.

Geometrically Riemann sums use rectangles of width Δx and height $f(z_i)$ on each subinterval. The function evaluation points, z_i , may be left endpoints, right endpoints, midpoints, minima, maxima, or random points in the i^{th} interval.

In the diagrams below, $\Delta x = [x_{i+1} - x_i]$ and z_i varies in value.



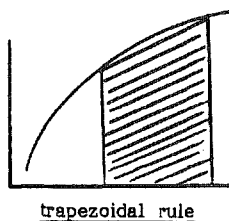
I N T E G R A T I O N

BACKGROUND INFORMATION (Continued)...

The trapezoidal rule uses trapezoids on each subinterval. Again using subintervals of equal length, an approximation sum is

$$\frac{1}{2} \left[f(x_0) + 2f(x_1) + 2f(x_2) + \dots + 2f(x_{n-1}) + f(x_n) \right] \Delta x$$

where each trapezoid has area $\frac{1}{2} [f(x_{i-1}) + f(x_i)] \Delta x$ and $\Delta x = \frac{b-a}{n}$.

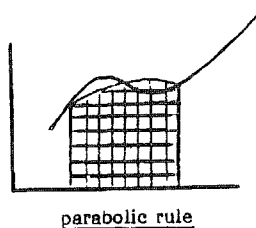


The parabolic rule (Simpson's rule) uses parabolas to approximate the curve over each pair of subintervals. The approximation sum is

$$\frac{1}{3} \left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \dots + 4f(x_{n-1}) + f(x_n) \right] \Delta x$$

where for each pair of subintervals the area is given by

$$\frac{1}{3} \left[f(x_{i-1}) + 4f(x_i) + f(x_{i+1}) \right] \Delta x \text{ and } \Delta x = \frac{b-a}{n}.$$



BACKGROUND INFORMATION (Continued)...The Computer Method

The user may select any number of methods for a single run of the program.

For the Riemann sums the points at which the function values are determined may be right endpoints, left endpoints, midpoints, points at which the integrand achieves a minimum on the subinterval, points at which the integrand achieves a maximum on the subinterval, or random selection of points.

Maximum and minimum points for each subinterval are determined by a search of many points in the subinterval, hence if there are many subintervals, these options require considerable time to run. Random points are found by using the RND function of the computer.

The subintervals are of equal length. The user must select the number of subintervals to be considered (less than 10000). The parabolic rule can be used only when an even number of subintervals is requested.

I N T E G R A T I O N

USE IN AN INSTRUCTIONAL SETTING...

Preparation

Students should understand the concept of integration and should be familiar with the various methods of approximating definite integrals.

They should have access to a copy of #19 - INTEGRATION Instructions and #20 - INTEGRATION Worked Example. A demonstration of the use of this program would be helpful to first-time users.

Using the Program

Students may wish to use the program GRAPH in conjunction with this program to plot the function they are examining.

At the beginning of the study of definite integrals, the student can explore Riemann sums for a variety of z_i point selection methods and different numbers of subintervals. This allows the student to gain an intuitive understanding of the definite integral as a limit and to gain an understanding of the role of the choice of z_i points in the definition.

During the study of the trapezoidal rule and the parabolic rule, the student can explore more examples and can consider examples with more subintervals than is possible with paper and pencil.

In the study of applications of integrals for which antiderivatives are not available or are difficult to attain, such as arc length, probability theory, etc. the student can concentrate on the application and yet see the final results of integration.

In numerical analysis, comparisons can be made with alternative methods.

INTEGRATION INSTRUCTIONS

INTEGRATION will compute numerical integration values for any continuous function of one variable on a finite interval.

- Step 1. Enter the function of x whose integral is to be approximated.
(remember to enter it using BASIC notation.)
- Step 2. Enter the interval of integration.
(left endpoint and right endpoint)
- Step 3. Specify the number of subintervals you wish to use.
- Step 4. Enter your choices of integration methods.

You may select as many of the following methods of numerical integration as you wish.

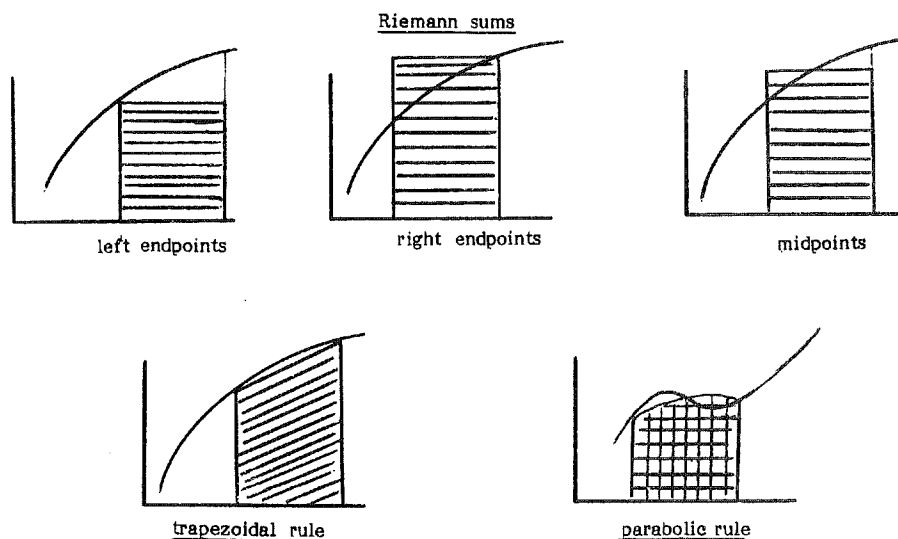
- 1) RECTANGLES - choices of function evaluation points are:

- | | |
|--------------------|-----------|
| A) LEFT ENDPOINTS | D) MINIMA |
| B) RIGHT ENDPOINTS | E) MAXIMA |
| C) MIDPOINTS | F) RANDOM |

- 2) THE TRAPEZOIDAL RULE

- 3) THE PARABOLIC RULE

Caution: Minima and maxima methods require considerable time to compute.



INTEGRATION WORKED EXAMPLE

Problem: Evaluate

$$\int_0^{\pi} \sin(x) dx$$

ENTER THE FUNCTION OF X WHOSE INTEGRAL IS TO BE APPROXIMATED.

Y = SIN(X)

Y = SIN(X)

WHAT IS THE INTERVAL OF INTEGRATION?

LEFT ENDPOINT: X = ? 0

RIGHT ENDPOINT: X = ? 3.14159265

HOW MANY SUBINTERVALS WOULD YOU LIKE ? 10

Y = SIN(X)

ON THE INTERVAL (0, 3.14159265)

WITH 10 SUBINTERVALS OF LENGTH .31415926

WHICH METHODS WOULD YOU LIKE TO USE TO APPROXIMATE THE INTEGRAL ?

- 1 - LEFT ENDPOINTS
- 2 - MIDPOINTS
- 3 - RIGHT ENDPOINTS
- 4 - MINIMA
- 5 - MAXIMA
- 6 - RANDOM POINTS
- 7 - TRAPEZOIDAL RULE
- 8 - PARABOLIC RULE

ENTER THE NUMBERS OF YOUR CHOICE. YOU MAY CHOOSE AS MANY OPTIONS AS YOU LIKE.

YOUR CHOICE ? 1, 2, 3, 4, 5, 6, 7, 8

METHOD	RESULT
LEFT ENDPOINTS	1.98352354
MIDPOINTS	1.98352354
RIGHT ENDPOINTS	2.00824841
MINIMA	1.66936427
MAXIMA	2.2976828
RANDOM POINTS	1.99863155
TRAPEZOIDAL	1.98352354
PARABOLIC	2.00010952

WOULD YOU LIKE TO

- 1. TRY DIFFERENT ENDPOINTS OR A DIFFERENT NUMBER OF SUBINTERVALS
- 2. TRY A DIFFERENT EQUATION
- 3. END

(ENTER 1, 2, OR 3) ?

INTEGRATION APPLICATIONS

1. Use the INTEGRATION program to calculate the value of

$$\int_0^1 \frac{e^{-x}}{1 - e^{-1}} dx$$

HINTS: Enter the function as $Y = \text{EXP}(-X)/(1 - \text{EXP}(-1))$.
Use 10 subintervals.

Do you think this integration problem might have an exact answer? What would it be? _____

2. Pollution from a factory is being released into a lake. Suppose the rate of concentration of the pollutant at time t is given by

$$P(t) = 140t^{5/2}$$

where t is the number of years since the factory started releasing the pollutants. Ecologists estimate that the lake can accept a pollution level of 4850 before all the fish life in the lake ends.

Can the factory continue dumping pollutants for 4 years without killing all the fish?

HINT: Is $\int_0^4 140t^{5/2} dt < 4850$? TYPE $140 * X^{(5/2)}$

3. A company wants to introduce a new machine which will produce a rate of annual savings given by $s(x) = 150 - x^2$, where x is the number of years of operation of the machine. However, the rate of annual cost is given by $c(x) = x^2 - (11/4)x$.

- For how many years will it be profitable to use the new machine?
- What are the total savings over the entire period of use of the machine?

HINT: Graph the two functions on the same axes. Determine their point of intersection. Find the area between the two curves, from zero to the point of intersection, using INTEGRATION.

INTEGRATION ANSWERS

1.	LEFT ENDPOINTS	1.0508332
	RIGHT ENDPOINTS	.950833195
	MID POINTS	.999583455
	RANDOM	.968913294
	TRAPEZOIDAL	1.0083319
	PARABOLIC	1.00000056

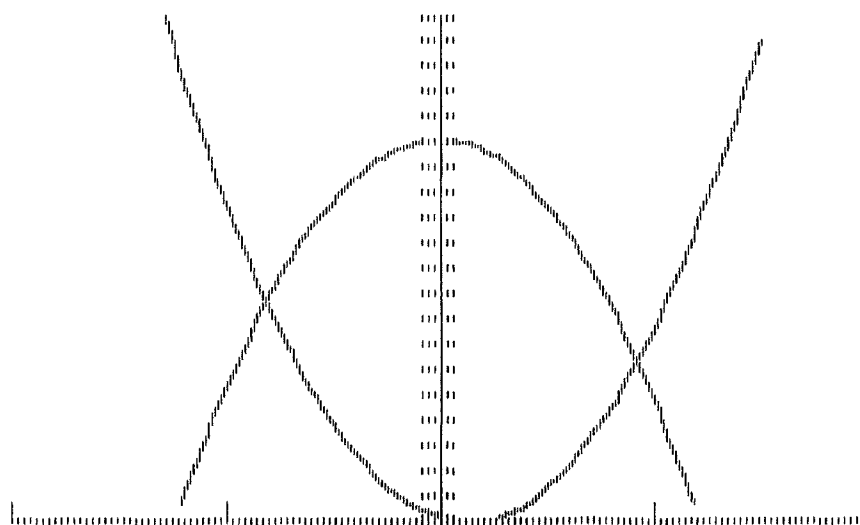
The exact answer is 1.

2. NO!
With 10 subintervals.

LEFT ENDPOINTS	4261.38
RIGHT ENDPOINTS	6053.38
MIDPOINTS	5101.30

RANDOM	5681.06
TRAPEZOIDAL	5157.38
PARABOLIC	5119.90

3. a) 8 years
b) about \$771



HOR: -20 TO 20 SCALE = 10
VER: 0 TO 200 SCALE = 10

INTEGRATION

SAMPLE RUNS

ENTER THE FUNCTION OF X WHOSE INTEGRAL
IS TO BE APPROXIMATED.

$$Y = X^2$$

The user enters the
function to be
integrated.

EXAMPLES OF SCREEN OUTPUT

WHAT IS THE INTERVAL OF INTEGRATION?

LEFT ENDPOINT: $X = 0$

RIGHT ENDPOINT: $X = 1$

HOW MANY SUBINTERVALS

WOULD YOU LIKE?

The interval is entered
and the number of
subintervals is chosen.

INTEGRATION

SAMPLE RUNS

```

      Y = X^2
      ON THE INTERVAL (0, 1)
      WITH 10 SUBINTERVALS OF LENGTH .1
-----
WHICH METHODS WOULD YOU LIKE TO USE
TO APPROXIMATE THE INTEGRAL?
- 1. LEFT ENDPOINTS
- 2. RIGHT ENDPOINTS
- 3. MIDPOINTS
- 4. MINIMA
- 5. MAXIMA
- 6. RANDOM
- 7. TRAPEZOIDAL
- 8. PARABOLIC

ENTER THE NUMBERS OF YOUR CHOICE. YOU
MAY CHOOSE AS MANY OPTIONS AS YOU LIKE.
YOUR CHOICE? 1,2,3,6,7,8,

```

The user specifies which methods will be used to compute the approximations.

EXAMPLES OF SCREEN OUTPUT

```

      Y = X^2
      ON THE INTERVAL (0, 1)
      WITH 10 SUBINTERVALS OF LENGTH .1
-----
METHOD                      RESULT
-----
LEFT ENDPOINTS               .285
RIGHT ENDPOINTS              .385
MIDPOINTS                    .3325

RANDOM                        .364634685
TRAPEZOIDAL                  .335
PARABOLIC                    .333333334

```

The approximations are printed in a table for each comparison among the methods used.

M U L T I - I N T E G R A L S

NUMERICAL INTEGRATION - SEVERAL VARIABLES

Specific Topic: Calculus
 Type: Algorithm Development and Problem Solving
Reading Level: 9-10 (Dale-Chall)
 Grade Level: 12 - Higher Education

DESCRIPTION...

This program will obtain a numerical approximation to a double or triple integral in a Cartesian, polar, cylindrical, or spherical system.

OBJECTIVES...

1. to obtain an approximation to a definite double or triple integral.
2. to use multiple integration to find the center of mass.
3. to use multiple integration to calculate volume.
4. to use multiple integration to determine the moment of inertia.

BACKGROUND INFORMATION...

The Mathematical Model

The double iterated integral for a continuous function f

$$\int_a^b \int_{f_1(u)}^{f_2(u)} f(u,v) \, dv \, du$$

can be approximated by the sum

$$\sum_{i=1}^n \sum_{j=1}^n f(s_i, t_j) \Delta v_j \Delta u_i$$

where the subintervals are all of equal length and the points (s_i, t_j) are all chosen in a similar manner such as the midpoints of subintervals and centers of rectangles.

Similarly the triple integral of a continuous function f

$$\int_a^b \int_{f_3(u)}^{f_4(u)} \int_{f_1(u,v)}^{f_2(u,v)} f(u,v,w) \, dw \, dv \, du$$

can be approximated by the sum

$$\sum_{i=1}^m \sum_{j=1}^n \sum_{k=1}^l f(s_i, t_j, r_k) \Delta w_k \Delta v_j \Delta u_i$$

where the subintervals are all of equal length and the points (s_i, t_j, r_k) all chosen in a similar manner such as midpoints of subintervals and centers of cubes.

BACKGROUND INFORMATION (Continued)...

The Computer Method

All functions must be entered using BASIC notation with constants expressed in decimal form. (See handout #1 - Mathematical Expressions in BASIC.)

For Cartesian systems use variables X,Y or X,Y,Z;
for polar systems (r,Θ) use variables R and THETA;
for cylindrical systems (r,Θ,z) use variables R, THETA, Z;
for spherical systems (ρ,Θ,Φ) use RHO, THETA, PHI.

There are 22 possible orders of integration from which the user must select.

The user must also select the subinterval size. Caution - a small subinterval size may require many subintervals and hence considerable time to compute.

For the double integral, the interval $[a,b]$ is partitioned according to the subinterval size selected. The midpoint s_i is selected in each subinterval and the interval

$[f_1(s_i), f_2(s_i)]$ is partitioned and the midpoints t_j selected.

Then the sum
$$\sum_{i=1}^m \left(\sum_{j=1}^n f(s_i, t_j) \Delta v \right) \Delta u$$
 is formed.

A similar iteration is used for the triple integral. In either case the resulting sum is printed as the numerical approximation of the integral.

M U L T I - I N T E G R A L S

USE IN AN INSTRUCTIONAL SETTING...

Preparation

Students need to understand the concept of multiple integrals. They should have had experience in approximating definite integrals of functions of one variable. The program INTEGRATION provides excellent preparation for MULTI-INTEGRALS.

The students should have copies of handout #22 - MULTI-INTEGRALS Instructions and #23 - MULTI-INTEGRALS Worked Example.

Using the Program

For an initial study of double or triple integrals, the student can gain some feeling for the limit concept by considering a sequence of subinterval sizes, such as $\{.5, .4, .3, .2, .1\}$.

The student can also change the order of integration and test the results.

Students also encounter many integrands that are not easily antidifferentiated or have no antiderivative. With this program such integrals can be approximated and the results used in a larger context.

Use of this program allows students to spend time on the concepts or applications being learned and quickly obtain the values of integrals. A multitude of time-consuming examples can be explored, e.g. finding a center of mass requires 4 integrals.

MULTI-INTEGRALS INSTRUCTIONS

MULTI-INTEGRALS performs double or triple iterated integration.

Step 1. Choose to compute either a double or triple integral.

Step 2. Select the coordinate system you want to use.

If you choose to compute a double integral, you can use the Cartesian or polar coordinate systems.

If you choose to compute a triple integral, you can use the Cartesian, cylindrical, or spherical coordinate systems.

Step 3. Enter the limits of integration and the integrand. The limits of integration and integrand can be either constants or functions. If you use functions, you must use the variable names indicated below.

<u>Coordinate System</u>	<u>Variable Names</u>
Cartesian	X, Y, Z
Polar	R, THETA
Cylindrical	R, THETA, Z
Spherical	RHO, THETA, PHI

Step 4. Choose the order of integration. The program will print out which orders of integration you may choose depending on the coordinate system used and whether you are computing a double or triple integral.

Step 5. Enter the subinterval size. The summation procedure used to approximate the integral will use the midpoints of the subintervals.

Caution: A small subinterval size, like .1, may require a considerable time to compute.

Step 6. After the numerical integration value has been computed, you may run the program again using a different size interval or a different integrand.

EXAMPLES

$$f(x,y) = x^2 + y^2$$

$$f(r,\theta) = r^3 \cos(\theta)$$

$$f(r,\theta,z) = \sqrt{1-r^2}$$

$$f(\rho,\theta,\phi) = \rho^2 \sin(\theta)$$

$$F(X,Y) = X^2 + Y^2$$

$$F(R,THETA) = R^3 * \cos(THETA)$$

$$F(R,THETA,Z) = \sqrt{1-R^2}$$

$$F(RHO,THETA,PHI) = (RHO)^2 * \sin(THETA)$$

MULTI-INTEGRALS WORKED EXAMPLE

Problem: Compute the value of

$$\int_0^1 \int_0^x (x + y + 1) dy dx.$$

ARE YOU USING A DOUBLE OR TRIPLE INTEGRAL ? D

DO YOU WANT TO USE

1) CARTESIAN COORDINATES

2) POLAR COORDINATES

ENTER 1 OR 2 ? 1

ENTER LOWER LIMIT OF INNERMOST INTEGRAL AS A CONSTANT OR A FUNCTION OF X AND Y.

F1 = 0

ENTER UPPER LIMIT OF INNERMOST INTEGRAL

F2 = X

ENTER LOWER LIMIT OF OUTER INTEGRAL AS A CONSTANT

F3 = 0

ENTER UPPER LIMIT OF OUTER INTEGRAL AS A CONSTANT

F4 = 1

ENTER INTEGRAND AS A CONSTANT OR A FUNCTION OF X AND Y

F = X + Y + 1

PLEASE CHOOSE THE ORDER OF INTEGRATION

1) DX DY 2) DY DX

ENTER A NUMBER FROM 1 TO 2 ? 2

ENTER SIZE OF SUBINTERVALS ? .1

COMPUTING

NUMERICAL INTEGRATION VALUE = 1.38

DO YOU WANT TO

1) RUN THIS PROGRAM AGAIN USING A DIFFERENT SIZE INTERVAL

2) RUN THIS PROGRAM AGAIN USING A NEW INTEGRAND AND NEW LIMITS

3) RETURN TO MENU

ENTER 1, 2, OR 3 ? 1

PLEASE CHOOSE THE ORDER OF INTEGRATION

1) DX DY 2) DY DX

ENTER A NUMBER FROM 1 TO 2 ? 2

ENTER SIZE OF SUBINTERVALS ? .05

COMPUTING

NUMERICAL INTEGRATION VALUE = 1.1825

MULTI-INTEGRALS APPLICATIONS

1. Use double integration to find the area bounded by the parabola $y = x^2$ and the line $y = x + 2$. Then plot both curves using the program GRAPH.



$$A = \int_{-1}^2 \int_{x^2}^{x+2} dy dx$$

2. A thin plate of uniform thickness and density covers the region of the xy-plane described in problem 1 above. Find its moment of inertia I_y about the y-axis by evaluating

$$\delta \int_{-1}^2 \int_{x^2}^{x+2} x^2 dy dx \quad \text{where } \delta \text{ is density.}$$

3. Find the moment of inertia, about the y-axis, of the area enclosed by the cardioid

$$r = 3(1 - \cos\theta). \quad (\text{Assume the density is 1.})$$

$$I_y = \int_0^2 \int_0^{3(1-\cos\theta)} r^3 \cos^2\theta dr d\theta$$

$$(\text{derived from the formula } I_y = \int_A x^2 dA.)$$

$$\text{Use } F = R^3 * \cos(\text{THETA}) * \cos(\text{THETA})$$

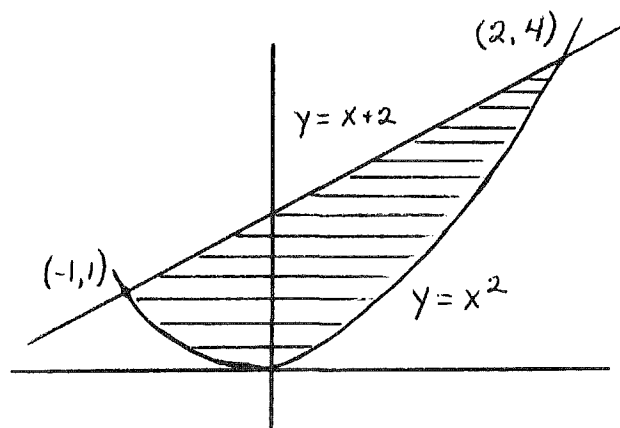
3. Use triple integration to find the volume bounded by the elliptic paraboloids $x = x^2 + 9y^2$ and $z = 18 - x^2 - 9y^2$

$$V = \int_{-3}^3 \int_{\frac{-\sqrt{9-x^2}}{3}}^{\frac{+\sqrt{9-x^2}}{3}} \int_{x^2+9y^2}^{18-x^2-9y^2} dz dy dx.$$

(Use subintervals of .5 of larger at first)

MULTI-INTEGRALS ANSWERS

1.



For interval size .5,
A = 5.75

For interval size .25,
A = 5.1875

For interval size .1,
A = 4.73

For interval size .05,
A = 4.63

(The exact answer is 4.5)

2. For interval size .5, $I_y = 5.2188 \delta$

For interval size .25, $I_y = 4.033 \delta$

For interval size .1, $I_y = 3.4126 \delta$

(The exact answer is $\frac{63}{20} \delta$)

3. For interval size .5, $I_y = 554$

For interval size .25, $I_y = 469$

For interval size .1, $I_y = 418$

(The exact answer is $49 \pi / 32$)

4. For interval size .5, $V = 93.625$

For interval size .25, $V = 88.297$

(The exact answer is 27π)

M U L T I - I N T E G R A L S

SAMPLE RUNS

ARE YOU USING A DOUBLE OR TRIPLE
INTEGRAL?DOUBLE

DO YOU WANT TO USE

1) CARTESIAN COORDINATES

2) POLAR COORDINATES

ENTER 1 OR 2? 1

The user enters each piece of required information as the program requests it.

EXAMPLES OF SCREEN OUTPUT

ENTER LOWER LIMIT OF INNERMOST INTEGRAL
AS A CONSTANT OR A FUNCTION OF
X AND Y.

$F1 = X^2$

ENTER UPPER LIMIT OF INNERMOST INTEGRAL
AS A CONSTANT OR A FUNCTION OF
X AND Y.

$F2 = X+2$

The limits of integration are entered for each of the two or three integrals. Then the function is entered.

MULTI-INTEGRALS

SAMPLE RUNS

PLEASE CHOOSE THE ORDER OF INTEGRATION

1> DX DY 2> DY DX

ENTER A NUMBER FROM 1 TO 2? 2

ENTER SIZE OF SUBINTERVALS? .5

NUMERICAL INTEGRATION VALUE = 5.21875

PRESS  TO CONTINUE

The user enters the order of integration and the subinterval size. The word "computing" flashes while the computations are being made. The value is printed on the screen.

EXAMPLES OF SCREEN OUTPUT

DO YOU WANT TO

1> RUN THIS PROGRAM AGAIN USING A
DIFFERENT SIZE INTERVAL

2> RUN THIS PROGRAM AGAIN USING A
NEW INTEGRAND AND NEW LIMITS

3> RETURN TO MENU

ENTER 1, 2, OR 3?

The user can then run the program again using a different size interval (without re-entering the other data) or can choose to do a different integration problem.

MULTI-INTEGRALS

SAMPLE RUNS

PLEASE CHOOSE THE ORDER OF INTEGRATION

1> DX DY 2> DY DX

ENTER A NUMBER FROM 1 TO 2? 2

ENTER SIZE OF SUBINTERVALS? .5

NUMERICAL INTEGRATION VALUE = 5.21875

PRESS ~~STOP~~ TO CONTINUE

The user enters the order of integration and the subinterval size. The word "computing" flashes while the computations are being made. The value is printed on the screen.

EXAMPLES OF SCREEN OUTPUT

DO YOU WANT TO

1> RUN THIS PROGRAM AGAIN USING A
DIFFERENT SIZE INTERVAL

2> RUN THIS PROGRAM AGAIN USING A
NEW INTEGRAND AND NEW LIMITS

3> RETURN TO MENU

ENTER 1, 2, OR 3?

The user can then run the program again using a different size interval (without re-entering the other data) or can choose to do a different integration problem.

LEAST SQUARES

LEAST SQUARES TECHNIQUE OF APPROXIMATING A FUNCTION

Specific Topic: Numerical analysis, statistics, calculus, finite mathematics,
linear algebra

Type: Problem Solving

Reading Level: 9-10 (Dale-Chall)

Grade Level: Higher Education

DESCRIPTION...

The program determines a least squares approximation of a power function, an exponential function, or a polynomial function to a set of data pairs. The data can be entered from the computer keyboard or from a data file on diskette. The errors of estimate can be listed and the standard error of estimate determined. A scattergram of the data can be plotted in high resolution graphics with the least squares function over it.

OBJECTIVES...

1. to determine a function that best fits a given set of data according to the least squares method.
2. to compare several approximations to a set of data, considering differences in errors of estimate.
3. to observe an application of partial derivatives.
4. to observe an example of the use of systems of linear equations.

LEAST SQUARES

BACKGROUND INFORMATION...

The Mathematical Method

The method of least squares is a technique of fitting a function f to a set of data pairs $(x_1, y_1), \dots, (x_n, y_n)$.

The method minimizes the sum of square errors

$$s = e_1^2 + \dots + e_n^2, \text{ where } e_i = y_i - f(x_i).$$

The function f may be a linear function, a non-linear polynomial, a power function, or an exponential function.

1. For the linear function $y = mx + b$, let $s = \sum_{i=1}^n (y_i - mx_i - b)^2$.

To minimize s , consider the partial derivatives set to zero, i.e. $\partial s / \partial m = 0$ and $\partial s / \partial b = 0$. The resulting equations are

$$\begin{aligned} nb + \left(\sum_{i=1}^n x_i \right) m &= \sum_{i=1}^n y_i \\ \left(\sum_{i=1}^n x_i \right) b + \left(\sum_{i=1}^n x_i^2 \right) m &= \sum_{i=1}^n x_i y_i \end{aligned}$$

2. For a polynomial of degree k , there are $k+1$ partial derivatives and hence $k+1$ equations in $k+1$ unknowns.

BACKGROUND INFORMATION (Continued)...

3. For a power function $y = ax^b$, there are two partial derivatives, and the resulting equations are

$$\begin{aligned} n \log a + \left(\sum_{i=1}^n \log x_i \right) b &= \sum_{i=1}^n \log y_i \\ \left(\sum_{i=1}^n \log x_i \right) \log a + \left(\sum_{i=1}^n (\log x_i)^2 \right) b &= \sum_{i=1}^n \log x_i \log y_i \end{aligned}$$

4. For an exponential function $y = ae^{bx}$, there are two partial derivatives, and the resulting equations are

$$\begin{aligned} n \log a + \left(\sum_{i=1}^n x_i \right) b &= \sum_{i=1}^n \log y_i \\ \left(\sum_{i=1}^n x_i \right) \log a + \left(\sum_{i=1}^n x_i^2 \right) b &= \left(\sum_{i=1}^n x_i \right) \log y_i . \end{aligned}$$

LEAST SQUARES

BACKGROUND INFORMATION (Continued)...

Various methods can be used to solve these equations. LEAST SQUARES uses the matrix method of row reduction. The matrix representation is always of the form $A^t A U = A^t B$, where

$$A = \begin{bmatrix} 1 & x_1 & \dots & x_1^k \\ 1 & x_2 & \dots & x_2^k \\ \vdots & \vdots & & \vdots \\ 1 & x_n & \dots & x_n^k \end{bmatrix}$$

for polynomials or the exponential function.

For the power function, x_i^j is replaced by $\log x_i^j$ in matrix A.

$$B = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

for polynomials

$$B = \begin{bmatrix} \log y_1 \\ \log y_2 \\ \vdots \\ \log y_n \end{bmatrix}$$

for exponential
or power functions,

$$\text{and } U = \begin{bmatrix} b \\ m \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} a \\ b \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} a_0 \\ \vdots \\ a_k \end{bmatrix}$$

The errors of estimate are e_1, e_2, \dots, e_n , and the standard error of estimate

is $\sum_{i=1}^n e_i^2 / (n - df)$, where df is the degrees of freedom.

LEAST SQUARES

BACKGROUND INFORMATION (Continued)...

The Computer Method

Data can be entered from the computer keyboard or from a sequential diskette file created by a previous run of the program.

The computer will print the list of the pairs if requested. The data pairs can be saved on the diskette. The errors of estimate, e_i , can also be listed and the standard error of estimate computed and printed.

LEAST SQUARES forms the matrices A , A^t , and B and then multiplies to get A^tA and A^tB . The augmented matrix is then formed and row reduced to echelon form using partial pivoting.

The program will display an error message and stop the calculation if logarithmic values are being used and the x_i or y_i values are not positive (greater than zero). This occurs only when using a power function.

If the scattergram option is requested, LEAST SQUARES plots the data points and then plots the function from the minimum x value to the maximum x value in high resolution graphics.

Examples

Data

(4, .01)	(10, .04)	(16, 1.24)
(5, .02)	(11, .09)	(17, 1.48)
(6, .02)	(12, .24)	(18, 1.73)
(7, .03)	(13, .38)	(19, 2.07)
(8, .03)	(14, .63)	(20, 2.50)
(9, .04)	(15, .93)	(21, 3.12)
		(22, 3.48)

exponential function

$$y = .0024829 e^{.356x}, \text{ standard error of estimate} = .61147$$

polynomial function of degree 2

$$y = .0166 x^2 - .246x + .851, \text{ standard error of estimate} = 5.571E-03$$

LEAST SQUARES

USE IN AN INSTRUCTIONAL SETTING...

Preparation

Students should understand the least squares method. They should either see a demonstration of this program or have a copy of handout #26 - LEAST SQUARES Worked Example. Handout #25 - LEAST SQUARES Instructions may be needed when using the program for the first time.

Students may wish to use their own diskette for this program in order to save files of data to be used on future runs.

The instructor may choose to create a data file to be used by the students when running the program.

Note that if data is to be saved (written) on the diskette, the diskette cannot be write-protected. (A diskette is write-protected if the notch on the right side of the diskette is covered by a sticker.)

Using the Program

In finite mathematics, this program can be used as an example in the study of systems of linear equations.

In calculus of several variables, this program illustrates a use of partial derivatives. The equations in the model are derived by using partial derivatives.

In linear algebra, the program provides an example of a use of matrices.

In numerical analysis, the program can be used to compare several approximations to a single data set, considering the differences in errors of estimate and the standard error of estimate. It can also be used to compare other approximation techniques.

This program has many applications outside of mathematics courses. It would be useful in areas such as physics, physical chemistry, engineering, biology, sociology, and economics to determine which mathematical functions best describe relationships and processes.

LEAST SQUARES INSTRUCTIONS

LEAST SQUARES will determine a least squares curve fit to a set of data pairs entered at the keyboard as you run the program or read from a diskette file created during an earlier run.

The curve can be

- 1) a polynomial of degree less than 10 (including a linear function)
- 2) a power function, $Y = A * X^B$, i.e. $y=ax^b$
- 3) an exponential function, $Y = A * EXP(B*X)$, i.e. $y=ae^{bx}$

- Step 1. Enter the type of curve desired.
- Step 2. Enter the degree of the curve if it is a polynomial.
- Step 3. Enter the source of the data (i.e., to be entered from the keyboard while running the program or to be read from a data file made during a previous run of the program.)
- Step 4. Enter the number of data points to be considered. (≤ 200)
- Step 5. Enter the data pairs from the keyboard or specify the file name of the data file on the diskette.
- Step 6. Ask to see the list of data pairs on the screen, if desired.
- Step 7. Change any data whgich was incorrectly entered or which is to be altered. (add, delete, or change)
- Step 8. Save the current data on the diskette if desired. If the data was read from a data file on the diskette, you do not need to save the data (it is already saved). Enter a name for the file. (Check that the diskette is not write protected; data cannot be saved on write-protected diskettes.)
- Step 9. Record the equation found by the computations.
- Step 10. At your request, the program will
- 1) list the errors of estimate, (one error of estimate for each data point)
 - 2) print the standard error of estimate,
 - 3) plot the curve over a scattergram of the datapoints.

LEAST SQUARES WORKED EXAMPLE

Problem: Find the straight line that best fits the points (0,1), (1,3), (2,2), (3,4), (4,5) according to the method of least squares.

DO YOU WANT TO USE A

- 1) POLYNOMIAL FUNCTION
- 2) POWER FUNCTION
- 3) EXPONENTIAL FUNCTION

ENTER 1, 2, OR 3 ? 1

ENTER DEGREE OF POLYNOMIAL (< 10) ? 1

FUNCTION TYPE: POLYNOMIAL OF DEGREE 1

DO YOU WANT TO READ YOUR DATA FROM

- 1) THE KEYBOARD
- 2) A DATA FILE

ENTER 1 OR 2 ? 1

ENTER NUMBER OF DATA POINTS? 5

ENTER PAIRS (X,Y) ONE PAIR PER LINE

(X1, Y1) ? 0,1
(X2, Y2) ? 1,3
(X3, Y3) ? 2,2
(X4, Y4) ? 3,4
(X5, Y5) ? 4,5

DO YOU WANT TO SEE A LIST OF THE DATA PAIRS (X,Y) ? N

DO YOU WANT TO CHANGE ANY DATA ? N

DO YOU WANT TO SAVE THE CURRENT DATA ON THE DISKETTE ? N

COMPUTING

THE POLYNOMIAL IS

$$Y = 0.9 X^1 + 1.2 X^0$$

(continued on next page)

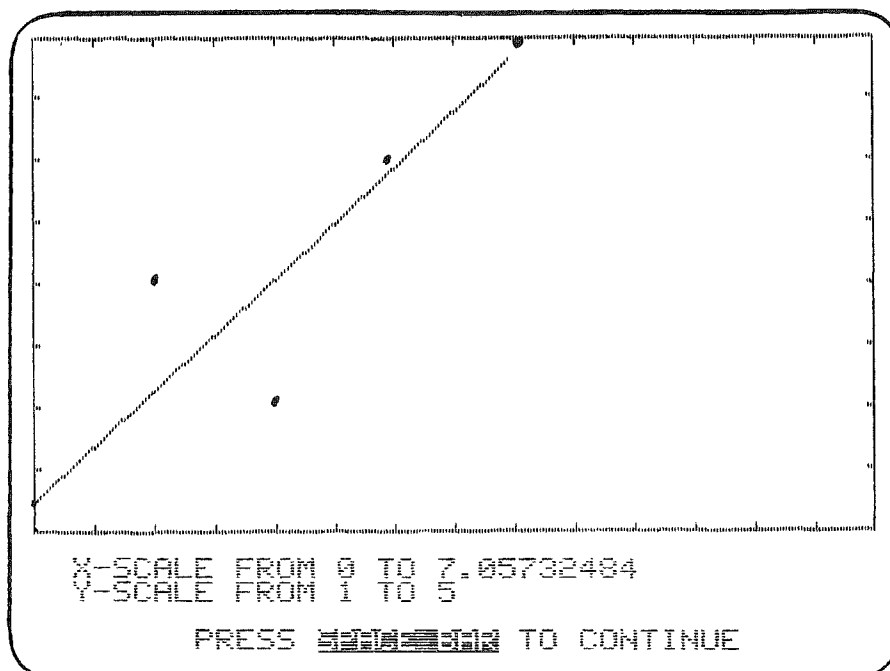
LEAST SQUARES WORKED EXAMPLE (Page 2)

DO YOU WANT THE ERRORS OF ESTIMATE ? Y

	<u>X</u> <u>VALUE</u>	<u>VALUE OF</u> <u>FN AT X</u>	<u>ERROR OF</u> <u>ESTIMATE</u>
1)	0	1.2	.199...
2)	1	2.1	.9
3)	2	3	1
4)	3	3.9	.099...
5)	4	4.8	.199...

STANDARD ERROR OF ESTIMATE IS 4.60800001

DO YOU WANT THE SCATTERGRAM WITH THE FUNCTION ? Y



Note: The first and last points will fall on the axes and may not be visible on the graph.

LEAST SQUARES APPLICATIONS

1. The vapor pressure of water at different temperatures is shown in the following table.

Temperature (°C)	0	20	40	60	80	100
Vapor pressure (torr=mmHg.)	4.6	17.5	55.3	149.4	355.1	760.0

Find the best possible least squares approximation to this data. Consider temperature the independent variable (x), vapor pressure the dependent variable (y). Try polynomials of degree 1, 2, and 3, and the exponential function. Use your findings to predict the vapor pressure at 90°C.

2. The solubility of copper sulphate in water ($\text{CuSO}_4 \cdot 5\text{H}_2\text{O}$) is given in the table below.

Temperature (°C)	0	10	20	30	40	50	60	70	80	90
Solubility (g/100 g solvent)	14.3	17.4	20.7	24.2	28.7	33.8	40.0	47.0	56.0	67.5

Find the best possible least squares approximation to this data. Consider a variety of functions. Use your findings to estimate the solubility of copper sulphate at 75°C.

3. A radioactive isotope of protactinium decays at the rate shown in the table below. The decay rate is measured at specific times with a geiger counter.

Time from start (sec)	0	40	80	120	160	200
Count rate	341	231	162	111	77	50

- a) Find the best possible least squares approximation to this data. Try the functions $y=x$, $y=x^2$, $y=e^x$, etc.
- b) EXTRA: Using your result from part a, calculate the half life of this isotope, that is, the time required for the decay rate to fall to one half its value. Use various initial values and compare the results (allowing for experimental error).

(Note: The power function cannot be used with a zero or negative x value).

LEAST SQUARES ANSWERS

1. Polynomial of degree 1 $y = 6.977x - 125.2$

Errors of Estimate
 129.8 3.2
 98.6 144.0
 77.9 187.5

Standard Error of Estimate
 14756.

- Polynomial of degree 2 $y = .117x^2 - 4.77x + 31.44$

Errors of Estimate
 26.8 34.5
 26.4 18.7
 46.5 30.9

Standard Error of Estimate
 746.67

- Polynomial of degree 3 $y = .0012x^3 - .06x^2 + 1.8x^2 + 2.6$

Errors of Estimate
 1.97 5.85
 3.68 4.35
 6.19 2.04

Standard Error of Estimate
 18.85

- Exponential function $y = 5.923 e^{.0508x}$

Errors of Estimate
 1.32 1.14
 10.11 24.58
 10.31 192.37

Standard Error of Estimate
 6303.4

2. Polynomial of degree 1 $y = .565x + 9.51$

Errors of Estimate
 4.8 2.2
 .12 2.3
 3.4 4.0
 3.4 2.1
 1.2 7.1

Standard Error of Estimate
 12.89

LEAST SQUARES ANSWERS (Page 2)

Polynomial of degree 2 $y = .00482x^2 + .131x + 15.3$

Errors of Estimate

Standard Error of Estimate

1.0	.3
.8	.6
.4	.1
.5	1.1
.7	1.3

.613

3. Exponential function $y = 342.4e^{-.0095x}$

Errors of Estimate

Standard Error of Estimate

1.4	3.3
1.6	1.2
1.9	1.4

3.79

Polynomial of degree 2 $y = .0062x^2 - 2.64x + 335.6$

Errors of Estimate

Standard Error of Estimate

5.4	8.7
1.7	3.5
6.0	4.5

29.35

LEAST SQUARES

SAMPLE RUNS

```
DO YOU WANT TO USE A
  1) POLYNOMIAL FUNCTION
  2) POWER FUNCTION
  3) EXPONENTIAL FUNCTION
ENTER 1, 2, OR 3? 1
ENTER DEGREE OF POLYNOMIAL ( <10 )?
```

The user specifies the type of function to be fitted to the data.

Use a polynomial of degree 1 for a straight line function.

EXAMPLES OF SCREEN OUTPUT

```
FUNCTION TYPE: POLYNOMIAL OF DEGREE 1
```

```
DO YOU WANT TO READ YOUR DATA FROM
  1) THE KEYBOARD
  2) A DATA FILE
ENTER 1 OR 2? 1
```

Data can be entered at the keyboard. The user can store data on the disk, choose a file name for the data file, then re-use the data by calling up that file name.

LEAST SQUARES

SAMPLE RUNS

FUNCTION TYPE: POLYNOMIAL OF DEGREE 1
NUMBER OF DATA POINTS: 5

THE POLYNOMIAL IS

$$Y = .9 X^1 + 1.2 X^0$$






PRESS      TO CONTINUE

The function found by the least squares method is printed.

EXAMPLES OF SCREEN OUTPUT

FUNCTION TYPE: POLYNOMIAL OF DEGREE 1
NUMBER OF DATA POINTS: 5

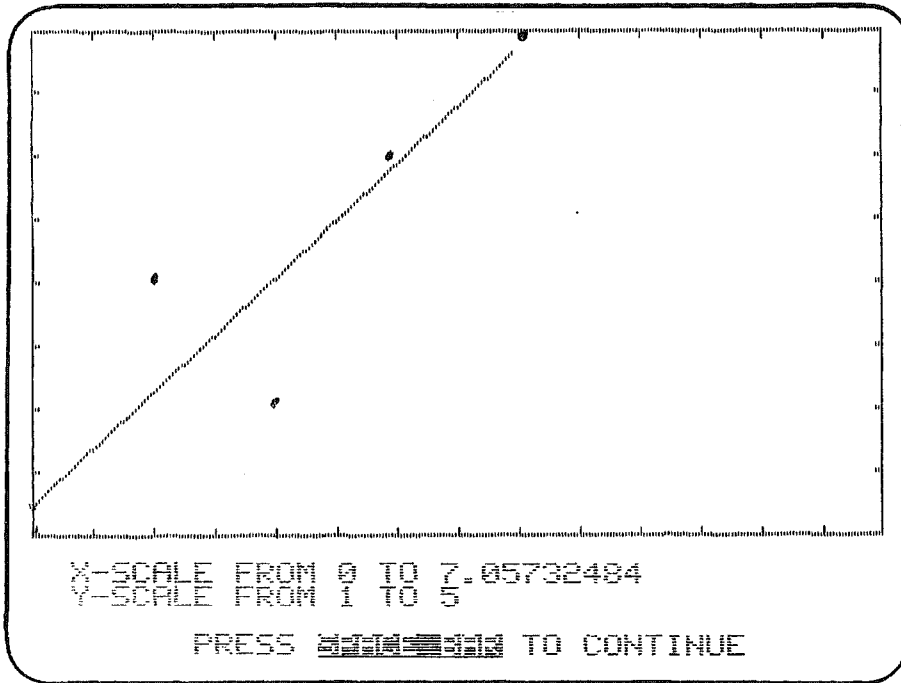
	X VALUE	VALUE OF FN AT X	ERROR OF ESTIMATE
1)	0	1.2	.1999999999
2)	1	2.1	.9
3)	2	3	1
4)	3	3.9	.09999999996
5)	4	4.8	.1999999999

PRESS      TO CONTINUE

The user can request the errors of estimate. Standard error of estimate is always printed.

LEAST SQUARES

SAMPLE RUNS



The user can request a scattergram of the data points and the curve fitting them.

EXAMPLE OF SCREEN OUTPUT

LINE GRAPH

GRAPHING SYSTEMS OF LINEAR EQUATIONS

Specific Topic: Beginning and intermediate algebra, college algebra, analytic geometry, finite mathematics

Type: Problem Solving

Reading Level: 9-10 (Dale-Chall)

Grade Level: 9 - Higher Education

DESCRIPTION...

The program will plot up to ten lines on one set of axes in high resolution graphics. After any line has been plotted, another line may be plotted on the same axes. The x interval and/or y interval of the graph may be changed and all the lines replotted.

OBJECTIVES...

1. to solve graphically, a system of equations in two variables.
2. to solve a system of inequalities in two variables.
3. to solve graphically, a linear programming problem.
4. to discover the relation between m in the equation $y = mx + b$ and the slope of the line that equation represents.
5. to relate the b in the equation $y = mx + b$ with the y-intercept of the corresponding graph.
6. to identify the common properties that exist in a family of equations representing parallel or perpendicular lines.

LINE GRAPH

BACKGROUND INFORMATION...

The user determines which part of the graph will appear on the screen by entering the x and y intervals. If the line does not cross the selected portion of the coordinate grid, a message informs the user that the line will not appear on the grid. The user may then select new x and y intervals to correct the situation.

Note that the scale is the same for the x and y axes, so it is usually best to have the x and y intervals about the same size. For example,

x: (-100,100) and y: (-5,5) is probably a poor choice, while
x: (-100,0) and y: (-80,0) is fine.

The equations of the lines must be of the form $x=b$ and $y=mx+b$, where m and b are integers or fractions expressed as p/q (with p and q integers and $q \neq 0$).

The user specifies whether the equation is of the form $x=b$ or $y=mx+b$, then enters the values for m and b. For example,

the equation $y=3$ is entered as $Y=+0X+3$;

the equation $y=-1/2x-1/4$ is entered as $Y=-1/2X-1/4$.

After each line is plotted, another line may be plotted or the x and y intervals may be changed. The program retains the values of m and b for each line, so that all lines are replotted after any interval change.

USE IN AN INSTRUCTIONAL SETTING...

Preparation

Students should be able to substitute values for x into a linear equation and determine the corresponding value of y . They should be able to plot ordered pairs as points on a rectangular coordinate system.

Using the Program

In analytic geometry the student can explore the visual effect of various slope values, m , and various intercept values, b .

The program can be used to estimate a point of intersection of two lines.

The program can be used to visualize a system of linear inequalities and determine the corners of the region.

In studying graphic solutions of linear programs in two variables, the constraints can be plotted. Then, either corners can be estimated and the corresponding values of the objective function determined, or the objective function can be plotted for various values to determine where it last contains points of the region.

LINE GRAPH INSTRUCTIONS

LINE GRAPH will plot up to 10 lines on a graph. You select the x-interval and y-interval for the coordinate axes.

Step 1. Enter the x-interval (x_1, x_2),
X1 =
X2 =

Enter the beginning and ending points of the x-interval, typing RETURN after each point.

Step 2. Enter the y-interval (y_1, y_2)
Y1 =
Y2 =

Step 3. Decide whether your equation is in the form of $x=b$ or $y=mx+b$.
Enter x or y.

Step 4. Enter the equation of your line.

The lines must be of the form
 $y = mx + b$ or $x = b$

where m and b must be integers or fractions

examples: $Y = +3X + 5/3$
 $Y = +0X-5$
 $Y = -2/3X - 5$
 $X = +5/4$

Enter the values for m and b, typing RETURN after each. Remember to use the negative symbol where necessary.

Step 5. After any line has been plotted, you may plot another line, change the x and y intervals and replot the lines, or start the program over with a new equation and new intervals.

LINE GRAPH WORKED EXAMPLE

Problem: The equation to be graphed is $y = -2x + 3$
 The x-interval is $(-2,4)$; y-interval is $(-1,5)$

ENTER X-INTERVAL (X1,X2).

X1 = -2

X2 = 4

ENTER Y-INTERVAL (Y1,Y2)

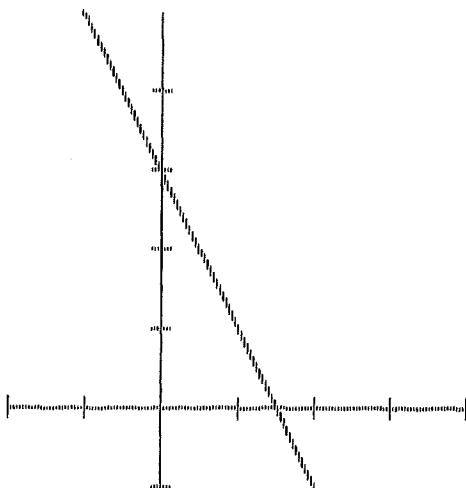
Y1 = -1

Y2 = 5

IS THE EQUATION IN THE FORM OF X= OR Y=
 (ENTER X OR Y)? **Y**

ENTER THE EQUATION OF YOUR LINE ?

Y = -2X+3



$Y = -2X + 3$

X: $(-2,4)$

SCALE = 1

Y: $(-1,5)$

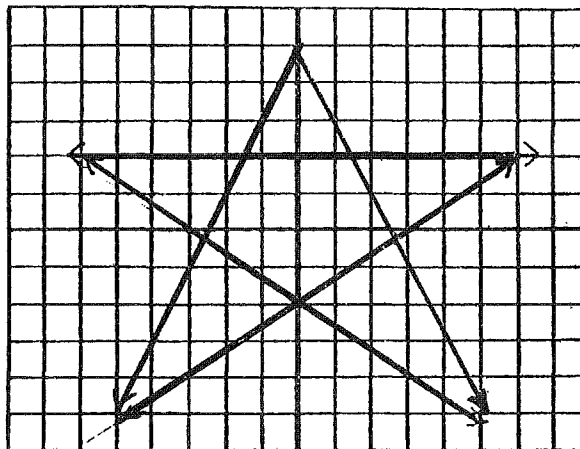
DO YOU WANT TO

1. PLOT ANOTHER LINE
2. CHANGE THE X AND/OR Y INTERVAL
3. START THE PROGRAM OVER
4. RETURN TO MENU

(ENTER 1, 2, 3, OR 4) ?

GRAPHING LINEAR EQUATIONS

1. Write equations of lines that will produce this figure. Verify your results by graphing the lines with LINE GRAPH.



2. Graph the region bounded by the lines

$$y = \frac{2}{3}x + 3$$

$$y = -\frac{1}{2}x + 4$$

$$y = -2x + 10$$

3. Graph the region represented by the inequalities

$$x + y \leq 9$$

$$-4x + 3y \leq 12$$

$$\frac{1}{2}x - y \leq -2$$

LINE GRAPH

SAMPLE RUNS

ENTER X-INTERVAL (X1,X2).

X1 = -2
X2 = 4

ENTER Y-INTERVAL (Y1,Y2).

Y1 = -1
Y2 = 5

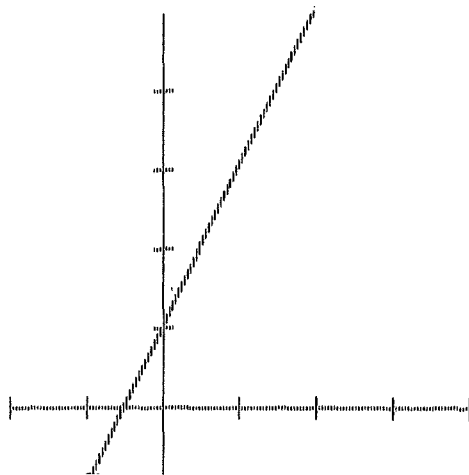
IS EQUATION IN THE FORM OF X= OR Y=
(ENTER X OR Y)? Y

ENTER THE EQUATION OF YOUR LINE

Y=+2X

The user enters the x and y intervals and the equation. The equation is entered by typing the values for m and/or b in $x=b$ or $y=mx+b$.

EXAMPLES OF SCREEN OUTPUT



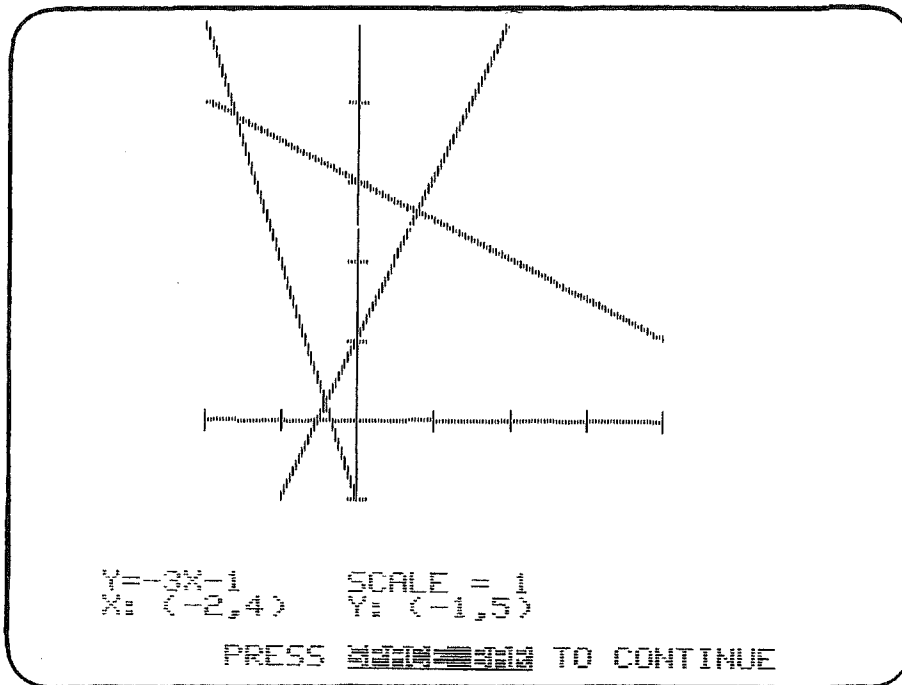
Y=+2X+1 SCALE = 1
X: (-2,4) Y: (-1,5)

PRESS  TO CONTINUE

The graph is plotted. Note that the axes are not numbered but the scale is given.

LINE GRAPH

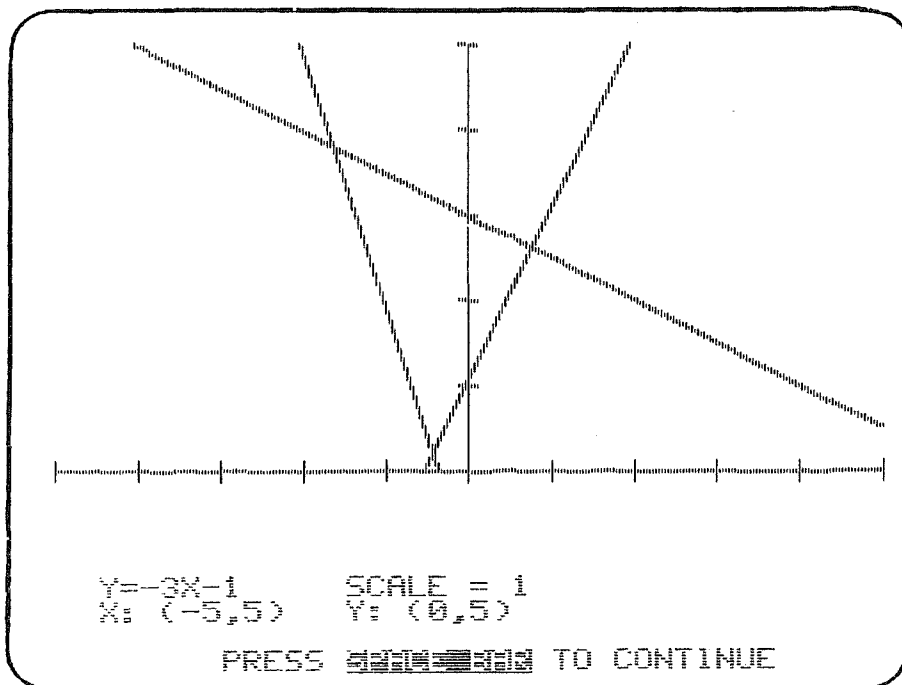
SAMPLE RUNS



Up to ten lines may be plotted on the same axes.

A set of lines may be graphed to be used in simple linear programming problems.

EXAMPLES OF SCREEN OUTPUT



The intervals may be changed and the lines automatically replotted.

ROW REDUCTION

ROW REDUCTION OF MATRICES

Specific Topic: College algebra, linear algebra, finite mathematics

Type: Problem Solving

Reading Level: 9-10 (Dale-Chall)

Grade Level: Higher Education

DESCRIPTION...

The program performs row operations on a matrix. The student determines the sequence of row operations needed to reduce the matrix. In this way, the student learns the algorithm, yet is spared the tedious arithmetical calculations.

OBJECTIVES...

1. to learn to select appropriate row operations to reduce a matrix.
2. to explore a wide variety of matrices.
3. to study the solution set for a system of equations with infinitely many solutions (as in the simplex method of linear programming).

R O W R E D U C T I O N

BACKGROUND INFORMATION...

Mathematical Method

The row operations which are used in the algorithm to row reduce a matrix are

- 1) interchange two rows,
- 2) multiply (or divide) a row by a nonzero constant, and
- 3) add a multiple of one row to another row.

By an appropriate choice of row operations any matrix can be row reduced to row-echelon form or to reduced row-echelon form.

A matrix is in row-echelon form if:

- 1) in any nonzero row the first nonzero entry is 1,
- 2) all zero rows are below all nonzero rows,
- 3) in any two successive nonzero rows, the leading 1 in a lower row occurs in a column to the right of the column of the leading 1 in the higher row.

A matrix is in reduced row-echelon form if additionally

- 4) each column containing a leading 1 has zeros elsewhere.

Computer Method

The row operations are

- 1) Add a multiple of one row to another: e.g. add -3 times row 5 to row 2 will change row 2.
- 2) Multiply a row by a constant: e.g. multiply row 2 by 5.
- 3) Divide a row by a constant: e.g. divide row 1 by -2.
- 4) Interchange of two rows: e.g. interchange row 2 and 3.

Note: To add one row to another, the multiple is 1; to subtract one row from another, the multiple is -1.

The maximum dimensions of the matrix are 9 by 9. The matrix elements must be less than 999 and greater than .01. The maximum number of digits printed in the matrix elements is 3, with not more than two digits to the right of the decimal point.

For example: 123.123 appears as 123., while .345 appears as .34, and 88.777 appears as 88.7.

ROW REDUCTION

USE IN AN INSTRUCTIONAL SETTING...

Preparation

The student needs a basic understanding of matrices and the meaning of row-echelon form of a matrix.

A class demonstration of this program and copies of handouts #30 - ROW REDUCTION Instructions and #31 - ROW REDUCTION Worked Example are helpful to first-time users.

Using the Program

In studying the process of row reducing matrices, a student will benefit from examining many examples. This program requires the student to select the row operation, and then the computer does the arithmetic, printing out the result. The time-consuming and error-prone arithmetic portion of the algorithm is done by the computer, but the decision process of choosing a row operation is left to the student. Hence the student concentrates on the algorithm that is being studied and not on arithmetic.

The student can explore a wide variety of examples, such as matrices with more rows than columns, matrices with more columns than rows, invertible matrices, and noninvertible square matrices.

If a matrix is invertible, it can be row reduced to the identity matrix, and the same sequence of reduction steps applied to the identity matrix will yield the inverse. The student can test this statement for invertible matrices.

The program can also be used to explore the solution set for a system of equations with infinitely many solutions. In particular, the program can be used to search for those solutions containing certain zeros, in preparation for the study of the simplex method in linear programming. For example,

the augmented matrix
$$\left[\begin{array}{ccccc|c} 1 & 5 & 0 & 0 & 4 & -1 \\ 0 & 0 & 1 & 0 & 6 & 1 \\ 0 & 0 & 0 & 1 & 2 & 4 \end{array} \right]$$
 has $(-1, 0, 1, 4, 0)$

as a solution. Row operations can change this to

$$\left[\begin{array}{ccccc|c} 1 & 5 & 0 & -2 & 0 & -9 \\ 0 & 0 & 1 & -3 & 0 & -11 \\ 0 & 0 & 0 & \frac{1}{2} & 1 & 2 \end{array} \right]$$
 and $(-9, 0, -11, 0, 2)$ shows as a solution.

ROW REDUCTION INSTRUCTIONS

ROW REDUCTION performs row operations to row reduce any matrix with less than 10 rows and less than 10 columns

- Step 1. Enter the matrix dimensions, the number of rows and the number of columns.
- Step 2. Enter the rows of the matrix, separating the entries with commas.
- Step 3. Select the row operation

The row operations are:

- 1) add a multiple of a row to another row
- 2) multiply a row by a constant
- 3) divide a row by a constant
- 4) interchange two rows

(Note: To add a row to another row, use the first option with 1 as the multiple. To subtract one row from another, use the first option with -1 as the multiple.)

Type information about row numbers and constants when the program requests it.

The program will perform the operation and print the new matrix.

This step may be repeated as many times as desired.

- Step 4. To end the program, choose option 5, END. If you need to make a handwritten copy of your final matrix, do so before typing 5, which clears the screen.

ROW REDUCTION WORKED EXAMPLE

Problem: Use row operations to reduce this matrix to the identity matrix.

$$\begin{bmatrix} 1 & 2 & 3 \\ 5 & 3 & 1 \\ 2 & 4 & 3 \end{bmatrix}$$

ENTER THE MATRIX DIMENSIONS:

HOW MANY ROWS? 3

COLUMNS? 3

TYPE IN A NUMBER FOR EACH OF THE 3 COLUMNS IN EACH ROW,
SEPARATING THE NUMBERS WITH COMMAS.

ROW 1 ? 1,2,3

ROW 2 ? 5,3,1

ROW 3 ? 2,4,3

1) 1 2 3

2) 5 3 1

3) 2 4 3

DO YOU WANT TO

1) ADD A MULTIPLE OF 1 ROW TO ANOTHER

2) MULTIPLY A ROW BY A CONSTANT

3) DIVIDE A ROW BY A CONSTANT

4) INTERCHANGE 2 ROWS

5) END

(ENTER 1-5)? 1

1) 1 2 3

2) 5 3 1

3) 2 4 3

ADD -5 TIMES ROW 1 TO ROW 2

PRESS SPACE BAR AFTER ENTERING EACH NUMBER. USE THE BACK
ARROW TO MAKE CORRECTIONS; ENTER 0 TO DO A DIFFERENT ROW
OPERATION.

1) 1 2 3

2) 0 -7 -14

3) 2 4 3

DO YOU WANT TO

1) ADD A MULTIPLE OF 1 ROW TO ANOTHER

2) MULTIPLY A ROW BY A CONSTANT

3) DIVIDE A ROW BY A CONSTANT

4) INTERCHANGE 2 ROWS

5) END

(ENTER 1-5)?

ROW REDUCTION APPLICATIONS

1. Find the row reduced form of the matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ .2 & .1 & .5 \\ 4.1 & 2.3 & 5.4 \end{bmatrix}$$

2. Find the inverse of the matrix

$$\begin{bmatrix} 2 & 1 & 3 \\ 3 & 2 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

Hint: The following list of row operations will reduce this matrix to the identity matrix. (Check this for yourself!) Then use these same row operations in the same order on the identity matrix, to obtain the inverse of the given matrix. The inverse matrix will contain decimal fractions.

Divide Row 1 by 2

Add -3 times Row 1 to Row 2

Add -1 times Row 1 to Row 3

Divide Row 2 by .5

Add 1.5 times Row 2 to Row 3

Add -.5 times Row 2 to Row 1

Divide Row 3 by -12

Add -5 times Row 3 to Row 1

Add +7 times Row 3 to Row 2

3. Find the inverse of the matrix (Hint: Use the procedure described in problem 2.)

$$\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

ROW REDUCTION ANSWERS

1.
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Identity Matrix

2.
$$\begin{bmatrix} -.08 & .25 & .42 \\ -.08 & .25 & -.58 \\ .42 & -.25 & -.08 \end{bmatrix}$$

3.
$$\begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$$

One possible set of operations

Add Row 1 to Row 2
Add Row 3 to Row 1
Divide Row 2 by 5
Add 2 times Row 2 to Row 3
Divide Row 3 by .2
Add Row 3 to Row 1
Add .4 times Row 3 to Row 2

ROW REDUCTION

SAMPLE RUNS

ENTER THE MATRIX DIMENSIONS:

HOW MANY ROWS? 3

COLUMNS? 3

The user enters the matrix dimensions, so that the computer can allot space to store the matrix.

EXAMPLES OF SCREEN OUTPUT

TYPE IN A NUMBER FOR EACH OF THE 3
COLUMNS IN EACH ROW, SEPARATING THE
NUMBERS WITH COMMAS.

ROW 1? 1,2,3

ROW 2? 4,5,6

ROW 3? -1,-3,-5

Then the user enters the matrix itself, row by row.

ROW REDUCTION

SAMPLE RUNS

```
1)  1   2   3
2)  4   5   6
3) -1  -3  -5
```

The array is printed and the user selects an option to begin to row reduce the matrix.

```
DO YOU WANT TO
1) ADD A MULTIPLE OF 1 ROW TO ANOTHER
2) MULTIPLY A ROW BY A CONSTANT
3) DIVIDE A ROW BY A CONSTANT
4) INTERCHANGE 2 ROWS
5) END
(ENTER 1 - 5)?
```

EXAMPLES OF SCREEN OUTPUT

```
1)  1   2   3
2)  4   5   6
3) -1  -3  -5
```

The user selects the multiples and the rows to be added. It may be useful to keep a record of the exact operations performed.

```
ADD ☐ TIMES ROW _ TO ROW _
```

```
PRESS SPACE BAR AFTER ENTERING EACH
NUMBER. USE THE BACK ARROW TO MAKE
CORRECTIONS; ENTER 0 TO DO A DIFFERENT
ROW OPERATION.
```

ROW REDUCTION

SAMPLE RUNS

```
1)      1      2      3
2)      4      5      6
3)      0     -1     -2
```

The new matrix is shown and the user is given another choice of operations.

```
DO YOU WANT TO
1) ADD A MULTIPLE OF 1 ROW TO ANOTHER
2) MULTIPLY A ROW BY A CONSTANT
3) DIVIDE A ROW BY A CONSTANT
4) INTERCHANGE 2 ROWS
5) END
(ENTER 1 - 5)?
```

EXAMPLES OF SCREEN OUTPUT

```
1)      1      0     -1
2)      0      1      2
3)      0      0      0
```

The user continues until the matrix is in row-echelon form.

```
DO YOU WANT TO
1) ADD A MULTIPLE OF 1 ROW TO ANOTHER
2) MULTIPLY A ROW BY A CONSTANT
3) DIVIDE A ROW BY A CONSTANT
4) INTERCHANGE 2 ROWS
5) END
(ENTER 1 - 5)?
```


MATRIX OPERATIONS

Specific Topic: Finite mathematics, linear algebra, matrix algebra

Type: Algorithm Development and Problem Solving

Reading Level: 9-10 (Dale-Chall)

Grade Level: Higher Education

DESCRIPTION...

The program performs operations on matrices of size 10 x 10 or smaller. The program uses diskette files to store up to 50 matrices so that the user need not enter them from the computer keyboard more than once.

The program provides 15 operations on matrices and an option to list them. The operations are:

1. input a matrix
2. store a matrix
3. obtain a stored matrix
4. print a matrix
5. transpose a matrix
6. add matrices
7. multiply a matrix by a scalar
8. multiply a matrix by another matrix on the left
9. multiply a matrix by another matrix on the right
10. obtain a power of a matrix
11. invert a matrix
12. row reduce a matrix
13. find the determinant of a matrix
14. find an orthonormal basis from the rows of a matrix
15. test a matrix for eigenvalues.

OBJECTIVES...

1. to perform any of the operations available with this program: transpose, invert, or row reduce a matrix, add or multiply matrices, find a determinant, an orthonormal basis, or find eigenvalues of a matrix.
2. to use matrices in a variety of applications.

BACKGROUND INFORMATION...

The Mathematical Methods

Inverse. If A is an invertible matrix, $n \times n$, and is augmented by I , $n \times n$, then after row reducing to echelon form, the last n columns of the augmented matrix contain the inverse of A , A^{-1} ; i.e., the matrix $A \ I$ row reduces to $I \ A^{-1}$.

Row reduced forms. Reduced row-echelon form has these properties:

1. the lead nonzero entry of any row is 1,
2. zero rows are below all nonzero rows
3. for two successive nonzero rows, the lead 1 of the lower row occurs in a column farther to the right than the column of the lead 1 of the upper row, and
4. if a column has a leading 1, all other entries are zero.

In this program, a matrix in row reduced form, has properties 1 and 4 above.

Determinants. The determinant of a matrix can be computed from the product of pivots, in the algorithm to row reduce a matrix. If there are an even number of row interchanges, the determinant is the product of pivots. If there are an odd number of row interchanges, the product of pivots is multiplied by -1 . If a matrix is not invertible, the determinant is zero.

Orthonormal basis. For the steps of the Gram-Schmidt orthogonalization process see any standard linear algebra text. An orthogonal basis can be converted to an orthonormal basis by normalizing the vectors of the basis.

Eigenvalues. λ is an eigenvalue of a matrix A if $(A - \lambda I)v = 0$ for nonzero eigenvectors v , i.e., if there exist nontrivial solutions to the system of equations represented by $(A - \lambda I)v = 0$. Hence $A - \lambda I$ is not invertible and then $\det(A - \lambda I) = 0$. (\det means determinant).

The Computer Method

MATRIX can be used to save and operate on up to 50 matrices of size 10×10 or smaller.

The matrices are stored in 10 arrays ARRAY S1,...,ARRAY S10 which are then stored on diskette for future recall in the current run of the program or some later run of the program.

The printing of a matrix requires a 14 character width for each column, hence printout on the monitor screen may be hard to read for matrices larger than 3×3 . A printer should be used with this program.

BACKGROUND INFORMATION (continued)

All operations on a matrix are done on the "current" matrix. Some operations change the current matrix, others do not.

The options are:

LIST prints a list of the numbered options

END terminates the program and saves all stored matrices on disk.

1. A matrix is input by the user. The input matrix becomes the new current matrix.
2. The current matrix is stored in a location selected by the user.
3. A previously stored matrix is obtained. The stored matrix becomes the new current matrix.
4. The current matrix is printed out on the printer and/or screen.
5. The current matrix is transposed. The transpose becomes the current matrix.
6. The current matrix is added to any matrix in storage. The sum becomes the new current matrix.
7. The current matrix is multiplied by a scalar. The product becomes the new current matrix.
8. The current matrix is left multiplied by another matrix in storage. The product becomes the new current matrix.
9. The current matrix is right multiplied by another matrix in storage. The product becomes the new current matrix.
10. The current matrix is multiplied by itself N times to obtain a power of the matrix. The power matrix becomes the new current matrix.
11. The current matrix is inverted. The inverse becomes the new current matrix.

BACKGROUND INFORMATION (continued)

12. The current matrix is row reduced. The current matrix can first be augmented for solving systems of 10 or fewer equations. The matrix can be put in row reduced form or in echelon form. The reduced form of the original matrix becomes the new current matrix and any augmented part is stored in a location of the user's choice. The reduced form does not interchange rows or columns in the search for a pivot but the information is retained for use in changing to echelon form if requested.
13. The determinant of the current matrix is computed and printed. The current matrix remains unchanged.
14. An orthonormal basis is obtained from the current matrix if the rows form a basis for a vector space. The orthonormal basis is the set of row vectors of the new current matrix.
15. For a current matrix A and successive values of X input by the user, the determinant of $A - XI$ is computed and printed. If the user enters 0 for an X value the program ends the option leaving $A - XI$ as the new current matrix.

Options 6, 8, 9, 10, 11, and 13 will not be performed if the matrices are not of the correct size. Then the current matrix remains unchanged. All row reducing is done by partial pivoting.

USE IN AN INSTRUCTIONAL SETTING...

Preparation

Students need to have a basic understanding of matrices and operations on matrices. They need to know the terminology associated with matrices.

They might first use the program ROW REDUCTION to learn how to row reduce matrices.

A class demonstration and copies of handouts #34 - MATRIX Instruction and #35 - MATRIX Worked Example are helpful to first-time users. If a printer is to be used as recommended, instruct the students in operating the printer.

Using the Program

In elementary courses such as finite mathematics most of the options can be used. If matrix operations are studied, the program ROW REDUCTION will give the student an understanding of row operations. This program can then be used to solve systems of linear equations.

In the study of Markov chains, this program can calculate power matrices quickly.

When studying the process of inverting matrices, students should first find inverses of matrices by using option 12 and augmenting the matrix with an identity matrix. Then when the algorithm is understood, option 11 can be used as a quicker method of obtaining the inverse.

In linear algebra the student should begin by using the program ROW REDUCTION to help with understanding the algorithm of row reducing a matrix. MATRIX can then be used throughout an elementary course. The program can be used in the study of topics such as elementary matrices, finding inverses by solving systems of equations, testing for linear dependence and independence, bases and dimension, change of basis, matrices of linear transformations, diagonalization, etc. In study of these topics students are often expected to set up and solve a system of linear equations, i.e., row reduce a matrix. The program becomes a tool for quick calculation of matrix operations so students can then concentrate on the more advanced concepts and topics.

This program can be of use in many other areas, such as the study of symmetry and quantum mechanics.

MATRIX INSTRUCTIONS

MATRIX can be used to store and operate on 50 or fewer matrices of size 10 x 10 or smaller. It is helpful to use a printer with this program.

There are 15 options for operating on a matrix. When the program requests an option, enter "list" if you want a list of the options, or "end" if you want to end the program.

The program needs space on your diskette to store 10 arrays (S1-S10). If the arrays do not exist on your diskette from a previous run, the program will create them. You may wish to save the diskette files containing your matrices to use at a later time. If you are not using a printer, you will want to keep a list of the diskette location of your matrices.

OPTIONS

- | | |
|----------------------|--|
| 1 - INPUT MATRIX | Enter a matrix at the keyboard. |
| 2 - STORE MATRIX | Save a matrix on diskette; specify the location. |
| 3 - OBTAIN MATRIX | Load from diskette into computer; stored matrix becomes current matrix. |
| 4 - PRINT MATRIX | Current matrix is printed on the screen (and printer if so specified). |
| 5 - TRANSPOSE MATRIX | The transpose becomes the current matrix. |
| 6 - ADD MATRICES | The sum of two matrices becomes the current matrix. |
| 7 - SCALAR MULTIPLY | The product of a scalar and a matrix becomes the current matrix. |
| 8 - LEFT MULTIPLY | The current matrix is left multiplied by a matrix stored on diskette. The product becomes the current matrix. |
| 9 - RIGHT MULTIPLY | The current matrix is right multiplied by a matrix stored on diskette. The product becomes the current matrix. |
| 10 - POWER | The current matrix is multiplied by itself N times, the power matrix becomes the current matrix. |
| 11 - INVERT MATRIX | The inverse of the matrix becomes the current matrix. |

MATRIX INSTRUCTIONS (Page 2)

- 12 - ROW REDUCE The current matrix is row reduced. It can be augmented for solving systems (≤ 10) of equations. Choose row reduced or echelon form. The reduced form of the matrix becomes the current matrix; any augmented part is stored in a new location.
- 13 - DETERMINANT The determinant is computed and printed; current matrix is unchanged.
- 14 - ORTHONORMAL BASIS Obtained if the rows form a basis for a vector space. It appears as the set of row vectors of the new current matrix.
- 15 - EIGENVALUES From current matrix, A, and values of X entered by the user, the determinant of $A - XI$ is computed and printed. When 0 is entered as X value, $A - XI$ is left as the current matrix.

To use this program, you first use option 1 to input one or more matrices unless you have matrices stored from a previous run. Then use option 2 to store matrices. Option 3 will make a stored matrix available for use. Option 4 will print the "current" matrix on the screen and the printer if so specified.

MATRIX INSTRUCTIONS (Page 3)

OPTION 5 - TRANSPOSE MATRIX

example:

$$\begin{array}{cc}
 \begin{bmatrix} 3 & 2 \\ 1 & 6 \\ 4 & 5 \\ 9 & 7 \end{bmatrix} & \begin{bmatrix} 3 & 1 & 4 & 9 \\ 2 & 6 & 5 & 7 \end{bmatrix} \\
 A & B, \text{ the transpose of } A
 \end{array}$$

Procedure

- 1 - input matrix A
- 2 - store in location 12 (if desired)
- 5 - transpose matrix, A (into B)
- 4 - print matrix, B

OPTION 7 - SCALAR MULTIPLY

example:

$$\begin{array}{ccc}
 1/2 \times \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} & = & \begin{bmatrix} .5 & 1 & 1.5 \\ 2 & 2.5 & 3 \\ 3.5 & 4 & 4.5 \end{bmatrix} \\
 \frac{1}{2} \times A & = & B
 \end{array}$$

Procedure

- 1 - input matrix A
- 7 - scalar multiply, enter the scalar as a decimal
- 4 - print matrix, B

OPTION 8 - LEFT MULTIPLY

example:

$$\begin{array}{ccc}
 \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} & \times & \begin{bmatrix} 10 & 11 & 12 \\ 13 & 14 & 15 \\ 16 & 17 & 18 \end{bmatrix} = \begin{bmatrix} 84 & 90 & 96 \\ 201 & 216 & 231 \\ 318 & 342 & 366 \end{bmatrix} \\
 A & \times & B = C
 \end{array}$$

Procedure

- 1 - input matrix A
- 2 - store in location 5
- 1 - input matrix B
- 2 - store in location 6
- 8 - left multiply current matrix, B, by matrix in location 5, A
- 4 - print matrix, C

MATRIX INSTRUCTIONS (Page 4)

OPTION 10 - POWER

example:
$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix} \quad \begin{bmatrix} 22 & 22 & 13 \\ 27 & 22 & 12 \\ 12 & 13 & 11 \end{bmatrix}$$

$A \qquad B = A \times A \times A = A^3$

Procedure

- 1 - input matrix A
- 10 - power, enter power (3)
- 4 - print matrix $B=A^3$

OPTION 11 - INVERT MATRIX

example:
$$\begin{bmatrix} 2 & -2 & 4 \\ 2 & 3 & 2 \\ -1 & 1 & -1 \end{bmatrix} \quad \begin{bmatrix} -.5 & .2 & -1.6 \\ 0 & .2 & .4 \\ .5 & 0 & 1 \end{bmatrix}$$

$A \qquad A^{-1}$

Procedure

- 1 - input matrix A
- 2 - store matrix A in location 6 (if desired)
- 11 - invert matrix A, (compute A^{-1})
- 4 - print matrix, A^{-1}
- 2 - store matrix, A^{-1} , in location 7 (if desired)

OPTION 12 - ROW REDUCE (See also examples #35)

example:
$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & -.999 & . . \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$A \qquad B$

Procedure

- 1 - input matrix, A
- 2 - store matrix, A (if desired)
- 12 - row reduce, matrix A
- 4 - print matrix, B

MATRIX INSTRUCTIONS (Page 5)

OPTION 13 - DETERMINANT

example:
$$\begin{vmatrix} 2 & 12 & 6 \\ 4 & 6 & 28 \\ 1 & 2 & 4 \end{vmatrix} = 60$$

A

Procedure

- 1 - input matrix, A
- 2 - store matrix, A, in location 4 (if desired)
- 13 - determinant

"DETERMINANT=60" is printed on the screen.

OPTION 14 - ORTHONORMAL BASIS

example:
$$\begin{bmatrix} 2 & -2 & 4 \\ 2 & 3 & 2 \\ -1 & 1 & -1 \end{bmatrix} \quad \begin{bmatrix} .408 & -.408 & .816 \\ .381 & .889 & .254 \\ -.829 & .207 & .518 \end{bmatrix}$$

A B

Procedure

- 1 - input matrix, A
- 2 - store matrix, A (if desired)
- 14 - orthonormal basis, B
- 4 - print matrix, B

MATRIX WORKED EXAMPLES

Addition of Matrices

Problem: Find $A + B$ where

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$$

WHICH OPTION: LIST, 1-15, END ? 1 (enter matrix)

ENTER THE MATRIX DIMENSIONS;

HOW MANY ROWS? 2

COLUMNS? 2

TYPE IN A NUMBER FOR EACH OF THE 2 COLUMNS IN EACH ROW,
SEPARATING THE NUMBERS WITH COMMAS.

ROW 1 ? 1,1

ROW 2 ? 0,1

WHICH OPTION: LIST, 1-15, END ? 2 (store matrix)

ENTER STORAGE LOCATION NUMBER ? 1

WHICH OPTION: LIST, 1-15, END ? 1 (enter matrix)

WHICH OPTION: LIST, 1-15, END ? 1 (enter matrix)

ENTER THE MATRIX DIMENSIONS;

HOW MANY ROWS? 2

COLUMNS? 2

TYPE IN A NUMBER FOR EACH OF THE 2 COLUMNS IN EACH ROW,
SEPARATING THE NUMBERS WITH COMMAS.

ROW 1 ? 1,3

ROW 2 ? 2,1

WHICH OPTION: LIST, 1-15, END ? 2 (store matrix)

ENTER STORAGE LOCATION NUMBER 2

WHICH OPTION: LIST, 1-15, END ? 6 (add matrices)

ENTER STORAGE NUMBER OF SECOND MATRIX ? 1

ADDITION COMPLETED

WHICH OPTION: LIST, 1-15, END ? 4 (print matrix)

2 4

2 2

Multiplication of Matrices

Problem: Find AB for matrices A and B above

WHICH OPTION: LIST, 1-15, END ? 3 (obtain matrix)

ENTER STORAGE LOCATION NUMBER ? 2 (matrix B)

WHICH OPTION: LIST, 1-15, END ? 8 (left multiply)

ENTER STORAGE LOCATION OF LEFT MATRIX ? 1 (matrix A)

MULTIPLICATION COMPLETED

WHICH OPTION: LIST, 1-15, END ? 4 (print matrix)

3 4

2 1

MATRIX WORKED EXAMPLES (Page 2)

Solving Systems of Equations Using Augmented Matrix

Problem: Solve $x-y+2z=9$ $x-y+2z=17$
 $x+z=15$ $x+z=-3$
 $2x-y+4z=17$ $2x-y+4z=-12$

WHICH OPTION: LIST, 1-15, END? 1

ENTER THE MATRIX DIMENSIONS:

HOW MANY ROWS? 3

COLUMNS? 5

TYPE IN A NUMBER FOR EACH OF THE 5 COLUMNS IN EACH ROW,
 SEPARATING THE NUMBERS WITH COMMAS.

ROW 1 ?1,-1,2,9,-7

ROW 2 ?1,0,1,5,-3

ROW 3 ?2,-1,4,17,-12

WHICH OPTION: LIST, 1-15, END? 12

DO YOU WANT TO AUGMENT THE MATRIX FOR
 SOLVING SYSTEMS OF EQUATIONS? NO

DO YOU WANT THE MATRIX

IN ECHELON FORM? YES

ROW REDUCTION COMPLETED

PRESS SPACE BAR TO CONTINUE

WHICH OPTION: LIST, 1-15, END? 4

1	0	0	2	-1
0	1	0	-1	2
0	0	1	3	-2

INTERPRETATION OF RESULTS

This is the row reduced echelon form of the augmented matrix. Hence the solutions are

System 1: $x=2, y=-1, z=3$

System 2: $x=-1, y=2, z=-2$

which appear in the last two columns of the matrix.

MATRIX WORKED EXAMPLES (Page 3)

Solving Systems of Equations Using The Coefficient Matrix A
and the Constant, Matrix, B

Problem:	Solve $x-y+2z=9$	$x-y+2z=-7$
	$x+z=5$	$x+z=-3$
	$2x-y+4z=17$	$2z-y+4z=-12$

WHICH OPTION: LIST, 1-15, END? 1

ENTER THE MATRIX DIMENSIONS:

HOW MANY ROWS? 3

COLUMNS? 3

TYPE IN A NUMBER FOR EACH OF THE 3 COLUMNS IN EACH ROW,
SEPARATING THE NUMBERS WITH COMMAS.

ROW 1 ?1,-1,2

(matrix of
coefficients)

ROW 2 ?1,0,1

ROW 3 ?2,-1,4

WHICH OPTION: LIST, 1-15, END? 12

DO YOU WANT TO AUGMENT THE MATRIX FOR

SOLVING SYSTEMS OF EQUATIONS? YES

ENTER NUMBER OF SYSTEMS? 2

ENTER CONSTANTS ONE ROW AT A TIME

ROW 1 ?9,-7

(matrix of
constants)

ROW 2 ?5,-3

ROW 3 ?17,-12

DO YOU WANT THE MATRIX

IN ECHELON FORM? YES

ENTER A STORAGE NUMBER FOR THE AUGMENTED

PART OF YOUR MATRIX? 3

THE REDUCED FORM OF THE AUGMENTED PART

IS STORED IN LOCATION NUMBER 3

(store augmented
part to print later)

AND THE REDUCED FORM OF THE ORIGINAL

MATRIX IS THE CURRENT MATRIX.

ROW REDUCTION COMPLETED

WHICH OPTION: LIST, 1-15, END? 4

(print reduced form
of coefficient
matrix)

1 0 0

0 1 0

0 0 1

WHICH OPTION: LIST, 1-15, END? 3

ENTER STORAGE LOCATION NUMBER? 3

(print reduced form
of constant matrix)

MATRIX LOADED FROM LOCATION 3

WHICH OPTION: LIST, 1-15, END? 4

2 -1

-1 2

3 -2

MATRIX WORKED EXAMPLES (Page 4)

INTERPRETATION OF RESULTS

The combined reduced form is

$$\begin{array}{ccccc} 1 & 0 & 0 & 2 & -1 \\ 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 1 & 3 & 2 \end{array}$$

from which we obtain the solutions,

System 1: $x=2, y=-1, z=3$

System 2: $x=-1, y=2, z=-2$

MATRIX APPLICATIONS

Use

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 4 & 0 \\ 2 & 1 \end{bmatrix}$$

1. Find $AB =$ $AC =$

$$CA =$$
 $CD =$

Store your results for use in problem 2.

2. Show that the associative law holds for

$$(AC)D = A(CD)$$

3. Give an example to show that the commutative law does not hold for matrix multiplication.

4. Find the inverse of matrices A and C

$$A^{-1} =$$

$$C^{-1} =$$

MATRIX APPLICATIONS (Page 2)

5. Using the program MATRIX and the matrix M given below, test the validity of the statement $MxM^{-1}=I$.

$$M = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 3 & 1 \\ 2 & 4 & 3 \end{bmatrix}$$

6. Solve this set of linear equations

$$\begin{aligned} 3x + 2y - z &= 4 \\ x + y + z &= 6 \\ x - 2y + 2z &= 3 \end{aligned}$$

using MATRIX (option 12).

7. Solve this set of linear equations

$$\begin{aligned} 2x_1 + x_2 + 3x_3 &= 1 \\ 3x_1 + 2x_2 + x_3 &= 6 \\ x_1 - x_2 &= -1 \end{aligned}$$

MATRIX ANSWERS

1. $AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$AC = \begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix}$

$CA = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$

$CD = \begin{bmatrix} 10 & 3 \\ 10 & 1 \end{bmatrix}$

2. Both matrices equal

$\begin{bmatrix} 20 & 4 \\ 10 & 1 \end{bmatrix}$

3. One might use $CA \neq AC$ as in problem 1 or many other examples.

4. $A^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

$C^{-1} = \begin{bmatrix} -.2 & .6 \\ .4 & -.2 \end{bmatrix}$

5. Compute M^{-1} , then left multiply by M (stored earlier on the diskette), to arrive at the identity matrix.

6. $\begin{bmatrix} 3 & 2 & -1 & 4 \\ 1 & 1 & 1 & 6 \\ 1 & -2 & 2 & 3 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$

The solution is $x=1$, $y=2$, $z=3$.

7. $X_1=1$, $X_2=2$, $X_3=-1$

M A T R I X

SAMPLE RUNS

ENTER THE MATRIX DIMENSIONS:

HOW MANY ROWS ?2
COLUMNS ?2■

The user enters the matrix size and the contents of the matrix.

EXAMPLES OF SCREEN OUTPUT

ENTER STORAGE LOCATION NUMBER? 1

To store this matrix, use option 2 (store) then assign a storage location number (e.g. 1). The matrix will then be stored on the disk.

CURRENT MATRIX STORED AT LOCATION 1

M A T R I X

SAMPLE RUNS

TYPE IN A NUMBER FOR EACH OF THE 2
COLUMNS IN EACH ROW, SEPARATING THE
NUMBERS WITH COMMAS.

ROW 1 ?1,3

ROW 2 ?2,1■

To add two matrices, use option 6 and give the storage numbers of the matrix to be added to the "current matrix". The current matrix is the last one used.

EXAMPLES OF SCREEN OUTPUT

ROW

COL

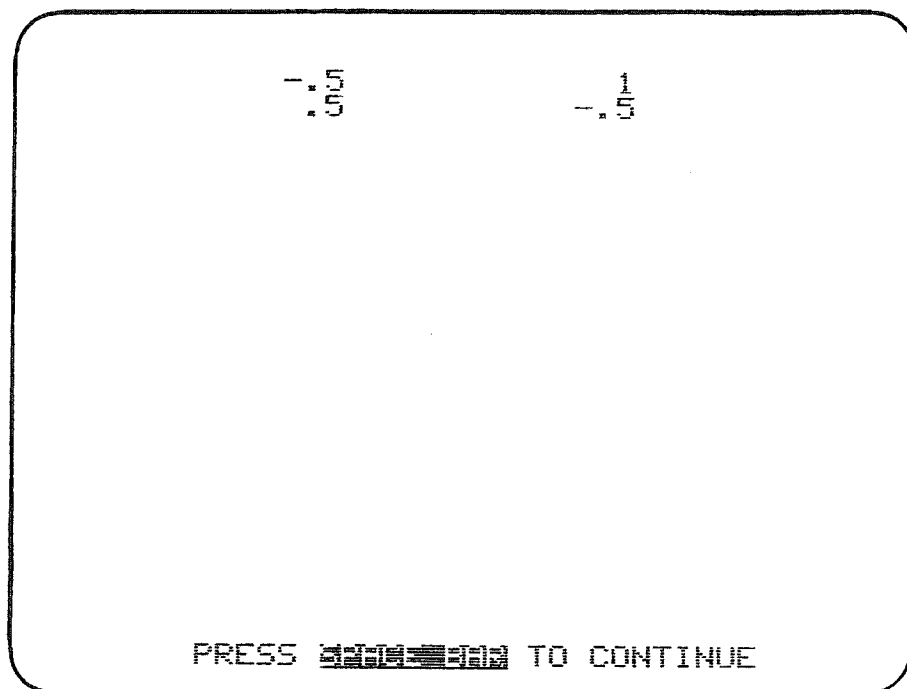
To see the matrix on the screen, use option 4 (print). This screen shows the sum of the current matrix and the matrix in storage location 1.

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$$

PRESS ■■■■■■ TO CONTINUE

M A T R I X

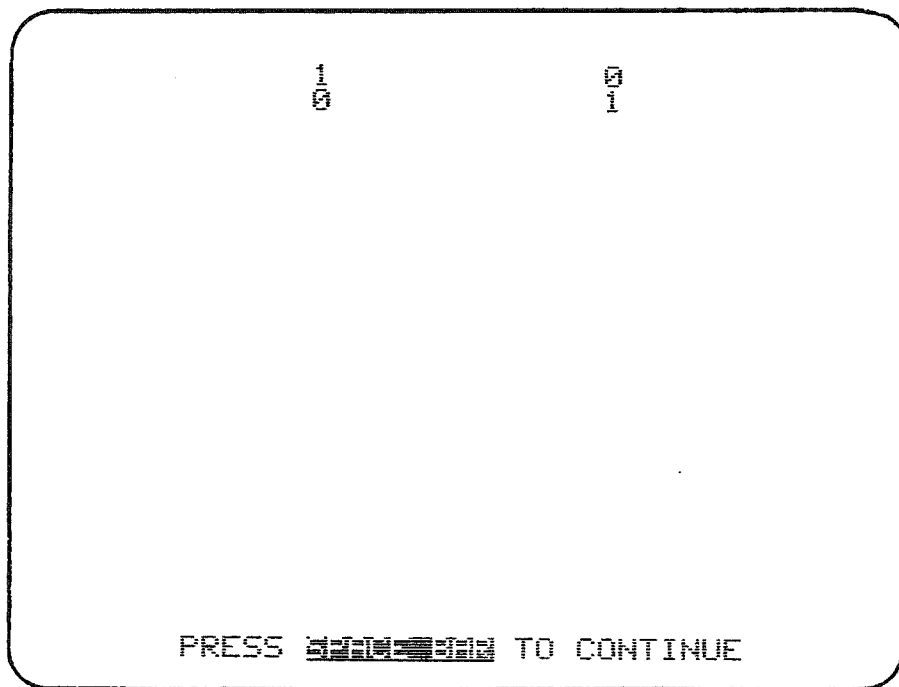
SAMPLE RUNS



To have a stored matrix become the "current matrix", use option 3 (obtain stored matrix).

To find the inverse of the current matrix, use option 11 (inverse), then option 4 (print).

EXAMPLES OF SCREEN OUTPUT



When option 12 (row reduce) is used, you may request the matrix in echelon form. This is the echelon form of matrix 2.

You may also augment the matrix for solving systems of equations.

SIMPLEX METHOD OF SOLVING LINEAR PROGRAMS _____

Specific Topic: Finite mathematics, linear algebra, operations research

Type: Problem Solving, Algorithm Development

Reading Level: 9-10 (Dale-Chall)

Grade Level: Higher Education

DESCRIPTION...

LINEAR PROGRAM 1 solves linear programs of the type: maximize $F = CX$, subject to $AX \leq B$, $X \geq 0$, and $B > 0$. The program uses the simplex method. The user must select the pivot row and pivot column at each step.

OBJECTIVES...

1. to select a correct pivot column and pivot row in a tableau representing a non-optimum solution.
2. to solve a linear program by the simplex method.

BACKGROUND INFORMATION...

The Mathematical Model

The standard form of a linear program is to maximize an objective function $f = c_1x_1 + \dots + c_nx_n$ subject to constraints of the form $a_1x_1 + \dots + a_nx_n \leq b$ where $b > 0$ and $x_i \geq 0$ for all i . In matrix form this becomes: maximize $F = CX$ subject to $AX \leq B$, $X \geq 0$, and $B > 0$. The simplex method converts these inequalities to equalities using slack variables. The system of inequalities becomes, in matrix form,

$$\begin{bmatrix} A & I \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = B$$

where I is an identity matrix and Y the matrix of slack variables.

The matrix $\begin{bmatrix} A & I & B \\ -C & 0 & 0 \end{bmatrix}$ is called the initial tableau.

For example, if the objective function is $f = 2x_1 + 3x_2$

with constraints $x_1 + 4x_2 \leq 7$ and $2x_1 + x_2 \leq 5$, then the initial tableau is

x_1	x_2	y_1	y_2	
1	4	1	0	7
2	1	0	1	5
-2	-3	0	0	0

BACKGROUND INFORMATION (continued)...

The first two rows represent a system of two equations in 4 unknowns which has infinitely many solutions. The obvious one from the tableau is (0,0,7,5) for which $f = 0$.

The simplex method searches that solution set for a solution giving the largest value for f . Changes in the tableau by row operations make other solutions obvious. The corresponding value for f is in the last row, last column.

See any finite mathematics text for the steps of the simplex method.

The Computer Method

The user must enter

1. the number of variables in the objective function
2. the coefficients of the variables in the objective function
3. the number of constraints
4. the coefficients of the variables (objective function) and the constants for each constraint. Note: Do not enter the slack variable coefficients.

The user must also enter the character width of the screen or printer receiving the output. Each column of the tableau requires a 12 character width. If the monitor or printer is used with a width insufficient to print the tableau as a single matrix, the program will print the tableau in sections. After each section is printed the user must hit the Space Bar to have the program continue and print the next section. This controlled pause allows the user to copy anything from the tableau that is needed before it scrolls off the screen.

The program will set up the slack variable part of the tableau and print the tableau. The user then selects a pivot column and pivot row for the row reduction to the next tableau. If the pivot entry is zero, the user is requested to try again. Any other poor choice of row or column will be processed and the user must determine the inadequacy of the step.

The user must also determine when a tableau represents the optimum solution.

LINEAR PROGRAM 1

USE IN AN INSTRUCTIONAL SETTING...

Preparation

The student should be familiar with the steps of row reducing. The program ROW REDUCTION should provide such experience. The student should also be familiar with the steps of the simplex method. This program will then allow the student to make all of the decisions in the steps of the simplex method, but have the computer do the arithmetic of the row reducing.

A class demonstration of this program and copies of handouts #37 - LINEAR PROGRAM 1 Instructions and #38 - LINEAR PROGRAM 1 Worked Examples will be helpful to first-time users.

Using the Program

The program can be used for standard form linear programs or programs for which the dual is in standard form.

It is helpful to have a printer connected to the computer when using this program, because of the large amounts of tabular data which make up the output.

LINEAR PROGRAM 1 INSTRUCTIONS

LINEAR PROGRAM1 will solve small maximum type linear programs that are written in standard form.

i.e., Programs of the type Maximize: $F = CX$

Subject to: $AX \leq B$, $X \geq 0$ and $B > 0$ (in matrix form)

- Step 1. Enter the number of variables in the objective function F.
- Step 2. Enter the coefficients for the x variables in the objective function F.
 (press the space bar after each coefficient)
- Step 3. Enter the number of constraints in $AX \leq B$, (0-19 constraints)
- Step 4. For each constraint, enter the constant and the coefficients (include zeros) of the constraint function.

The $X \geq 0$ constraints are not entered.

The program prints the first tableau in sections which depend on the width of your printer or screen. The objective function is listed with the negatives of the coefficients in the bottom row. The value of F is found in the bottom row last column.

After any tableau is printed, a new tableau can be obtained. You select the pivot column and pivot row. Hence you control the selection of tableaus and the termination of the program.

LINEAR PROGRAM 1 WORKED EXAMPLE

Problem: Maximize $f = 5x_1 - 2x_2 + x_3$ (objective function)

subject to $7x_1 - 2x_2 + x_3 \leq 8$
 $-x_1 + 3x_2 + 2x_3 \leq 4$ 3 constraints
 $3x_1 + x_2 + 6x_3 \leq 6$

NUMBER OF OBJECTIVE FUNCTION VARIABLES ? 3
 OBJECTIVE FUNCTION COEFFICIENTS ? $F = 5 X1 + -2 X2 + 1 X3$
 NUMBER OF CONSTRAINTS ? 3
 CONSTRAINT COEFFICIENT AND CONSTANT
 ENTER CONSTANT FOR CONSTRAINT 1 ? 8
 ENTER CONSTRAINT FUNCTION
 $7 X1 + -2 X2 + 1 X3 < =8$
 ENTER CONSTANT FOR CONSTRAINT 2 ? 4
 ENTER CONSTRAINT FUNCTION
 $-1 X1 + 3 X2 + 2 X3 < =4$
 ENTER CONSTANT FOR CONSTRAINT 3 ? 6
 ENTER CONSTRAINT FUNCTION
 $3 X1 + 1 X2 + 6 X3 < =6$
 ENTER PRINTER OR SCREEN WIDTH ? 40
 TABLEAU NUMBER 1

SECTION	1	2	3
	1		
	-7.000	-2.000	1.000
	-1.000	3.000	2.000
	3.000	1.000	6.000
	-5.000	2.000	-1.000

SECTION	2	5	6
	4		
	1.000	0.000	0.000
	0.000	1.000	0.000
	0.000	0.000	1.000
	0.000	0.000	0.000

SECTION	3
	7
	8.000
	4.000
	6.000
	0.000

DO YOU WANT A NEW TABLEAU? Y
 ENTER PIVOT COLUMN NUMBER 1
 ENTER PIVOT ROW NUMBER 1

LINEAR PROGRAM 1 WORKED EXAMPLE (Page 2)

TABLEAU NUMBER 2 (not shown)

DO YOU WANT A NEW TABLEAU ? Y

ENTER PIVOT COLUMN NUMBER 3

ENTER PIVOT ROW NUMBER 3

TABLEAU NUMBER 3

SECTION 1

1	2	3
1.000	-.333	0.000
0.000	2.000	0.000
0.000	.333	1.000
0.000	.666	0.000

SECTION 2

4	5	6
.153	0.000	-.025
.307	1.000	-.384
-.076	0.000	.179
.692	0.000	.051

SECTION 3

7
1.076
4.153
.461
5.864

INTERPRETATION OF RESULTS

$$f = 5.864, x_1 = 1.076, x_2 = 0, x_3 = .461$$

LINEAR PROGRAM 1 APPLICATIONS

A frozen food company processes potatoes into packages of French fries, hash browns, and flakes (for mashed potatoes). The raw potatoes are sorted by size and quality, and allocated to the three product lines.

The company buys the potatoes from two sources, which differ in sizes and quality. The table below shows the percentages of each source used for each product.

Product	Source 1	Source 2
French fries	20%	30%
Hash browns	20%	10%
Flakes	30%	30%
(Waste)	(30%)	(30%)

Suppose that the relative profit from the two sources is 5 and 6 respectively, and that the maximum number of each product that the company can sell is 1.8 (million pounds) of French fries, 1.2 (million pounds) of hash browns, and 2.4 (million pounds) of flakes.

The table below summarizes this information.

Product	Source 1	Source 2	Sales Limits
French fries	.2	.3	1.8
Hash browns	.2	.1	1.2
Flakes	.3	.3	2.4
Relative Profit	5	6	

We wish to maximize the profit,

$$f = 5x_1 + 6x_2$$

subject to

$$\begin{aligned} .2x_1 + .3x_2 &\leq 1.8 \\ .2x_1 + .1x_2 &\leq 1.2 \\ .3x_1 + .3x_2 &\leq 2.4 \end{aligned}$$

$$x_1 \geq 0 \text{ and } x_2 \geq 0.$$

Use LINEAR PROGRAM 1 to find the optimal solution.

LINEAR PROGRAM 1 ANSWERS

Initial Tableau

.2	.3	1	0	0	1.8
.2	.1	0	1	0	1.2
.3	.3	0	0	1	2.4
-5	-6	0	0	0	0

Pivot column 2, Pivot row 1.

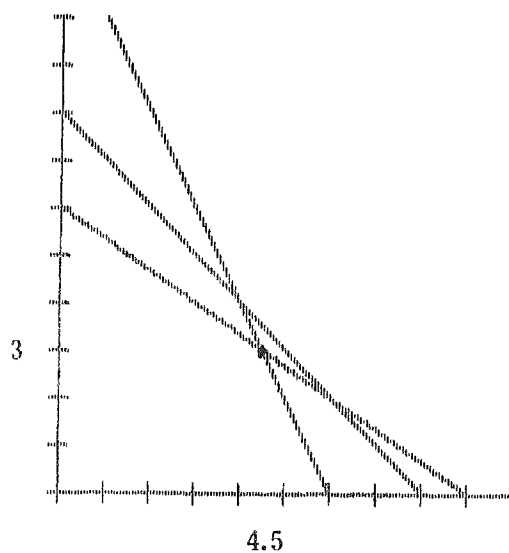
Tableau Number 2

.666	1	3.333	0	0	6
.133	0	-3.333	1	0	.6
.100	0	-1	0	1	.6
-1	0	20	0	0	36

Pivot column 1, Pivot row 2,

Tableau Number 3 (Final Tableau)

0	1	5	-5	0	3
1	0	-2.5	7.5	0	4.5
0	0	-.75	-.75	1	.15
0	0	17.5	7.5	0	40.5

Maximum profit is 40.5, when $x_1 = 4.5$ and $x_2 = 3$ 

LINEAR PROGRAM 1

SAMPLE RUNS

OBJECTIVE FUNCTION COEFFICIENTS

F = 5 X1 -2 X2 + 1 X3

The user enters the coefficients of the objective function, pressing the space bar after each coefficient is entered.

EXAMPLES OF SCREEN OUTPUT

CONSTRAINT COEFFICIENTS AND CONSTANT

ENTER CONSTRAINT FUNCTION

7 X1 -2 X2 + 1 X3 <= 8

Enter the number of constraints, then enter the first constraint constant and the coefficients of the first constraint. Continue this process for each constraint.

PRESS  TO CONTINUE

LINEAR PROGRAM 1

SAMPLE RUNS

TABLEAU NUMBER 1

SECTION 1

1	2	3
7.000	-2.000	1.000
-1.000	3.000	2.000
3.000	1.000	6.000
<hr/>		
-5.000	2.000	-1.000

SECTION 2

4	5	6
1.000	0.000	0.000
0.000	1.000	0.000
0.000	0.000	1.000
<hr/>		
0.000	0.000	0.000

PRESS ~~ENTER~~ TO CONTINUE

The first tableau is printed in a form that matches the width of your screen or printer.

EXAMPLES OF SCREEN OUTPUT

ENTER PIVOT COLUMN NUMBER? 1

ENTER PIVOT ROW NUMBER? 2

The user then can create a new tableau by specifying the pivot column and pivot row.

SIMPLEX METHOD OF SOLVING LINEAR PROGRAMS

Specific Topic: Finite mathematics, linear algebra, operations research

Type: Problem Solving

Reading Level: 9-10 (Dale-Chall)

Grade Level: Higher Education

DESCRIPTION...

LINEAR PROGRAM 2 will solve linear programs of the type

maximize $F = CX$

subject to $AX \leq B, B > 0, X \geq 0$

The program uses the simplex method and automatically selects the pivot row and pivot column necessary to compute the solution.

OBJECTIVES...

1. to solve a linear program in standard form by the simplex method.
2. to solve a linear program whose dual is in standard form by the simplex method.

BACKGROUND INFORMATION...

The Mathematical Method

The standard form of a linear program is to maximize an objective function $f = c_1x_1 + \dots + c_nx_n$ subject to constraints of the form $a_1x_1 + \dots + a_nx_n \leq b$ where $b > 0$ and $x_i \geq 0$ for all i . In matrix form this becomes: maximize $F = CX$ subject to $AX \leq B$, $X \geq 0$, and $B > 0$. The simplex method converts these inequalities to equalities using slack variables. The system of inequalities becomes in matrix form:

$$\begin{bmatrix} A & I \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = B$$

where I is an identity matrix and Y the matrix of slack variables.

The matrix $\begin{bmatrix} A & I & B \\ -C & 0 & 0 \end{bmatrix}$ is called the initial tableau.

For example, if the objective function is $f = 2x_1 + 3x_2$

with constraints $x_1 + 4x_2 \leq 7$ and $2x_1 + x_2 \leq 5$, then the initial tableau is

x_1	x_2	y_1	y_2	
1	4	1	0	7
2	1	0	1	5
-2	-3	0	0	0

BACKGROUND INFORMATION (Continued)...

The first two rows represent a system of two equations in 4 unknowns which has an infinite number of solutions. The obvious one from the tableau is (0,0,7,5) for which $f = 0$.

The simplex method searches that solution set for a solution giving the maximum value for f . The value of f is found in the last row, last column of the tableau.

See any finite mathematics text for steps of the simplex method.

The Computer Method

The user enters

1. the number of variables in the objective function
2. the coefficients of the variables in the objective function
3. the number of constraints
4. the coefficients of the variables (objective function) and the constants for each constraint. Note: Do not enter the slack variable coefficients.

The user also enters the character width of the screen or printer receiving the output. Each column of the tableau requires a 12 character width. If the monitor or printer is used with a width insufficient to print the tableau as a single matrix, the program will print the tableau in sections. After each section is printed the user must press the Space Bar to have the program continue and print the next section. This controlled pause allows the user to copy anything from the tableau that is needed before it scrolls off the screen.

The program will set up the slack variable part of the tableau and print the tableau.

Notice that in this program, the user does not specify the pivot column and pivot row, as is required in LINEAR PROGRAM 1.

LINEAR PROGRAM 2

USE IN AN INSTRUCTIONAL SETTING...

Preparation

The student should be familiar with the steps of the simplex method. The program LINEAR PROGRAM 1 should provide such experience.

Using the Program

This program allows the student to apply the simplex method to problems in standard form and to study problems whose dual is in standard form. The student can then concentrate on problem solving using linear programs, rather than arithmetic or matrix operations.

It is useful to have a printer connected to the computer when using this program.

LINEAR PROGRAM 2 INSTRUCTIONS

LINEAR PROGRAM2 will solve small maximum type linear programs that are written in standard form. i.e., programs of the type

maximize: $F = CX$

subject to: $AX \leq B$, $X \geq 0$ AND $B \geq 0$ (in matrix form)

- Step 1 Enter the number of variables in the objective function F,
- Step 2 Enter the coefficients for the x variables in the objective function
- Step 3 Enter the number of constraints in $AX \leq B$
- Step 4 For each constraint, enter the constant and the coefficients (including zeros) of the constraint function.

The $x \geq 0$ constraints are not entered.

The program will next print the first tableau in sections dependent on the width of your printer or screen. The objective function is listed with the negatives of the coefficients in the bottom row. The value of f is found in the bottom row last column. The program will then determine and print the final tableau using the simplex method.

LINEAR PROGRAM 2 WORKED EXAMPLE

Problem: maximize $t = x_1 + 1.2x_2 + 1.5x_3$

subject to $4x_1 + 3x_2 + 9x_3 \leq 160$

$3x_1 + 4x_2 + x_3 \leq 50$

$2x_1 + 4x_2 + 3x_3 \leq 50$

NUMBER OF OBJECTIVE FUNCTION VARIABLES ? 3

OBJECTIVE FUNCTION COEFFICIENTS ? $F = 1 X1 + 1.2 X2 + 1.5 X3$

NUMBER OF CONSTRAINTS ? 3

CONSTRAINT COEFFICIENTS AND CONSTRAINTS

ENTER CONSTANT FOR CONSTRAINT 1? 160

ENTER CONSTRAINT FUNCTION

$4 X1 + 3 X2 + 9 X3 \leq 160$

ENTER CONSTANT FOR CONSTRAINT 2? 50

ENTER CONSTRAINT FUNCTION

$3 X1 + 4 X2 + 1 X3 \leq 50$

ENTER CONSTANT FOR CONSTRAINT 3? 50

ENTER CONSTRAINT FUNCTION

$2 X1 + 4 X2 + 3 X3 \leq 50$

ENTER PRINTER OR SCREEN WIDTH ? 40

TABLEAU NUMBER 1

SECTION	1	2	3
	1		
	4.000	3.000	9.000
	3.000	4.000	1.000
	2.000	4.000	3.000
	-1.000	-1.250	-1.500

SECTION	2	5	6
	4		
	1.000	0.000	0.000
	0.000	1.000	0.000
	0.000	0.000	1.000
	0.000	0.000	0.000

SECTION	3
	7
	160.000
	50.000
	50.000
	0.000

LINEAR PROGRAM 2 WORKED EXAMPLE (Page 2)

TABLEAU NUMBER 2

SECTION 1

1	2	3
-2.000	-9.000	0.000
2.333	2.666	0.000
.666	1.333	1.000
<hr/> 0.000	<hr/> .800	<hr/> 0.000

SECTION 2

4	5	6
1.000	0.000	-3.000
0.000	1.000	-.333
0.000	0.000	.333
<hr/> 0.000	<hr/> 0.000	<hr/> .500

SECTION 3

7
10.000
33.333
16.666
<hr/> 25.000

INTERPRETATION OF RESULTS

The value of f is 25; $x_1=0$, $x_2=0$ and $x_3=.333$

LINEAR PROGRAM 2 APPLICATIONS

Suppose you have two assembly lines manufacturing three different items.

Type one items require 5 hours work on line 1 and 4 hours on line 2.

Type two items need 10 hours on line 2 only.

Type three items need 8 hours on line 1 only.

Your profit is as follows:

Type one items	-----	\$100 each
Type two items	-----	\$500 each
Type three items	-----	\$700 each

The assembly lines can run only 40 hours per week and the number of items made cannot be negative.

How should you run the assembly line to maximize your profit?

Let X_1 = number of type one items
 X_2 = number of type two items
 X_3 = number of type three items

Then $X_1 \geq 0$, $X_2 \geq 0$, $X_3 \geq 0$

$$\begin{array}{rcl} 5 * X_1 + 8 * X_3 & \leq & 40 \quad \text{for line 1} \\ 4 * X_1 + 10 * X_2 & \leq & 40 \quad \text{for line 2} \end{array}$$

We will maximize

$$P = 100 * X_1 + 500 * X_2 + 700 * X_3.$$

$$\begin{array}{rcl} 5 * X_1 + 0 * X_2 + 8 * X_3 & \leq & 40 \\ 4 * X_1 + 10 * X_2 + 0 * X_3 & \leq & 40 \end{array}$$

Use LINEAR PROGRAM 1 or 2 to find the solution.

LINEAR PROGRAM 2 ANSWERS

Maximize $F = 100x_1 + 500x_2 + 700x_3$

Subject to $5x_1 + 8x_3 \leq 40$
 $4x_1 + 10x_2 \leq 40$

Initial Tableau

5	0	8	1	0	40
4	10	0	0	1	40
-100	-500	-700	0	0	0

Final Tableau

.625	0	1	.125	0	5
.4	1	0	0	.1	4
537.5	0	0	87.5	50	5500

The maximum profit is \$5500, obtained when producing 0 type one items, 5 type two items, and 4 type three items.

LINEAR PROGRAM 2

SAMPLE RUNS

OBJECTIVE FUNCTION COEFFICIENTS

$$F = 1 X_1 + 1.2 X_2 + 1.5 X_3$$

The user enters the number of objective function variables and their coefficients, pressing the space bar between entries.


EXAMPLES OF SCREEN OUTPUT

CONSTRAINT COEFFICIENTS AND CONSTANT

ENTER CONSTRAINT FUNCTION

$$4 X_1 + 3 X_2 + 9 X_3 \leq 160$$

The user enters the constant and then the coefficients for each of the constraint inequalities.

PRESS  TO CONTINUE

LINEAR PROGRAM 2

SAMPLE RUNS

TABLEAU NUMBER 1

SECTION 1

1	2	3
4.000	3.000	9.000
3.000	4.000	1.000
2.000	4.000	3.000
<hr/>		
-1.000	-1.200	-1.500

SECTION 2

4	5	6
1.000	0.000	0.000
0.000	1.000	0.000
0.000	0.000	1.000
<hr/>		
0.000	0.000	0.000

PRESS  TO CONTINUE

Tableau number 1 is printed in a format appropriate to the user's printer or screen width.

EXAMPLES OF SCREEN OUTPUT

SECTION 2

4	5	6
.085	-.212	.148
-.170	.574	-.702
-.148	-.127	.489
<hr/>		
.119	.102	.108

SECTION 3

7	
1.063	
2.127	
10.638	
<hr/>	
16.489	

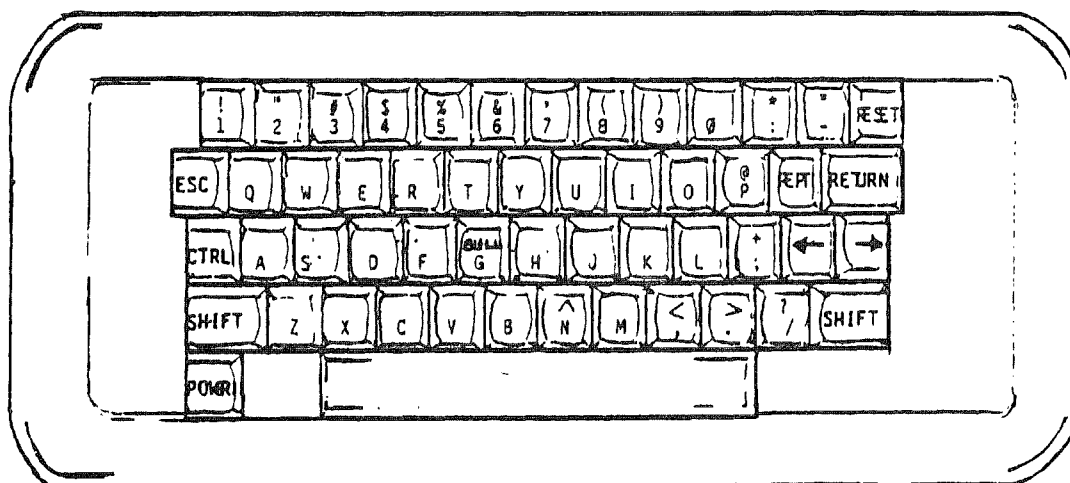
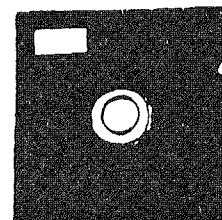
PRESS  TO CONTINUE

Next the final tableau is automatically computed and printed.

APPENDIXES

GETTING TO KNOW YOUR APPLEEquipment

- FLOPPY DISKETTE:** A 5¼ inch 'record' that contains a series of computer programs.
- DISK DRIVE:** A box unit that selects the program or data of your choice, displays the available list of programs on the diskette, and saves or deletes programs on the diskette.
- MONITOR OR TELEVISION:** Either a monitor or a television set can be used to display the information.

APPLE Keyboard

The APPLE keyboard looks much like the keyboard of a regular typewriter as far as placement of letters and numbers. A few of the special keys are noted below:




- SHIFT:** Shift key - located in the lower left or right-hand corner of the keyboard. When looking at the keys, you will notice two symbols on some keys. If you press the key, you will print the lower symbol. If you press the "SHIFT" button and a key, you will print the top symbol. For example: If you hold down the "SHIFT" key while you press the semicolon (;), you will get the plus (+) symbol printed.
- RESET:** Reset key - located in the upper right-hand corner. Press this key if something goes wrong or in order to work with a different diskette.

CTRL: Control key - located on the left side of the keyboard. In some of the instructions to the computer, you must hold the "CONTROL" button down while pressing another key. For example, the notation "CONTROL" C will be used which means to hold the "CONTROL" key down and press the "C" key simultaneously.

Note: Control characters do not appear on the screen.

RETURN: Return key - located below the "RESET" key. When you are finished typing a line, you normally press this key. In this manual, when you are to press the "RETURN" key, it will be noted by "RETURN".

SPACE BAR: The space bar is the long rectangular key at the bottom of the keyboard. On most programs, you will see at the bottom of the screen - PRESS SPACE BAR TO CONTINUE. At that point you would use the space bar key. In general, use the return key after you type information into the computer. Use the space bar to proceed from one screen to another.

BACKSPACE: Backspace key - located under the "REPT" key. The "  " key will be used to erase mistakes. Each time you press the "  " key, the flashing white square called the "cursor" will move back one space. To correct mistakes, press the "  " until you locate the mistake, make the correction, and continue typing the line.

POWER: Power light - located on the bottom of the keyboard. This key says "POWER" but is only a light to show if the system is on. It cannot turn on the system. (The ON-OFF system is on the back left side.)

Zero - located on the top row of the keys is used for the number zero. Capital "O" is the letter "O". (In this manual zeros do not have slashes through them.)

WORKING WITH THE APPLE

1. TURN on the television (or monitor).
2. INSERT A DISKETTE IN THE DISK DRIVE (Exposed oval part is inserted first with the diskette label up.)
3. Close the door on the disk drive.
4. Ground yourself by touching the metal plate near the on-off switch on the back left side of the APPLE.
5. Turn on the APPLE.
6. "APPLE II" appears on the screen and the disk drive light turns on. You will hear a whirling sound from the disk drive. If the disk drive light does not go off in about 10 seconds, turn the APPLE off and make sure your diskette is placed correctly in the disk drive.
7. A State of Minnesota map appears on the screen with the diskette name. Then a "menu" appears. The menu gives you a list of programs on the diskette. The program named DESCRIPTIONS shows you descriptions of the other programs on the diskette, then returns to the menu. The program END allows you to insert and use a different diskette. Type the number of the program you want to use. Press the RETURN key after you type your request.
8. Follow the directions given in the program. Remember to press the RETURN key after each input. Press the SPACE BAR upon request.
9. If you want to start over before the end of the program or see a new diskette, do either of the following:
 - a. Press the RESET key, insert proper diskette and type PR#6.
 - b. Turn the APPLE off, insert proper diskette, then turn it on again.

TURNING OFF THE COMPUTER

1. Take the diskette out of the disk drive.
2. Turn off the APPLE.
3. Turn off the television (or monitor).

Note: Diskettes are sensitive to dust, heat, cold, and magnetic fields. Handle them with care.

DEFINITIONS OF TERMS

BACKGROUND INFORMATION - Information which explains or enriches content or provides technical information on the program.

COURSEWARE - A collection of computer programs together with accompanying support materials.

DOCUMENTATION - Written material for the teacher to use with the computer program (Also called a support booklet).

DRILL AND PRACTICE - A type of computer program which acts as drillmaster by providing repetitive practice on a skill or set of facts.

EDUCATIONAL GAME - A type of computer program with an instructional purpose presented in a game format.

GRADE LEVEL - The grade level at which the program fits into the curriculum.

HANDOUTS - Pages of the support booklet which are to be duplicated for student or teacher use.

LESSON PLAN - A day-by-day guide to implement classroom use of the program including ideas on organization, class projects, and follow-up activities.

MODULE - The package which contains both the microcomputer programs and the support booklet.

OBJECTIVES - The results to be achieved by using the program and materials as outlined.

PROBLEM SOLVER - A type of computer program which processes data for a student defined problem.

PROGRAM - The routines and operations which instruct the computer.

READING LEVEL - The readability of the text that appears on the computer screen.

SAMPLE RUNS - Pages of the support booklet with examples of computer screen output and accompanying explanations to outline the flow of the program.

SELO - Some Essential Learner Outcomes prepared by the Minnesota State Department of Education. When applicable these are included with the objectives in MECC support booklets.

SIMULATION - A type of computer program which approximates a real-world environment for examination.

SUPPORT BOOKLET - Written material which provides the information a teacher may need to use the program in a classroom situation (also called documentation).

TEACHER AID - A type of computer program to assist a teacher with classroom management tasks.

TUTORIAL - A type of computer program which provides new information as well as repetitive drill and practice in teaching a concept.

MATHEMATICS VOLUME 4

CREDITS

GRAPH

The original program, POLYGRAPH, was authored by Darrell Ricke, MECC. Modifications were made by Beverly Durkee, Augsburg College.

FUNCTION LIMITS

The program was authored by Beverly Durkee, Augsburg College, with additional programming by Todd Bailey, MECC.

MATRIX

The program was authored by Beverly Durkee, Augsburg College, with additional programming by Rick Crist, MECC.

POLYROOT

NEWTON 1

NEWTON 2

INTEGRATION

MULTI-INTEGRALS

LEAST SQUARES

LINE GRAPH

ROW REDUCTION

LINEAR PROGRAM 1

LINEAR PROGRAM 2

These programs were authored by Beverly Durkee, Augsburg College.

MATHEMATICS VOLUME 4**TECHNICAL INFORMATION**GRAPH

Main Program: GRAPH
 Chains to: GRINST

FUNCTION LIMITS

Main Program: LIMFCTN
 Binary File: TAPA
 Address: 5480
 Length: 10970

POLYROOT

Main Program: POLYRT

NEWTON 1

Main Program: NEWT1
 Binary File: TAPA
 Address: 5480
 Length: 10970

NEWTON 2

Main Program: NEWT2
 Binary File: TAPA
 Address: 5480
 Length: 10970

INTEGRATION

Main Program: NUMINT
 Binary File: TAPA
 Address: 5480
 Length: 10970

MULTI-INTEGRALS

Main Program: MULTINT
 Binary File: TAPA
 Address: 5480
 Length: 10970

LEAST SQUARES

Main Program: LSTSQR
 Chains to: SCAT

LINE GRAPH

Main Program: LINGRAPH

ROW REDUCTION

Main Program: ROWRED

MATRIX

Main Program: LINEAR

LINEAR PROGRAM 1
Main Program: LPCODE1

LINEAR PROGRAM 2
Main Program: LPCODE2

MECC INSTRUCTIONAL SERVICES ACTIVITIES

PURPOSE: The primary purpose of the Minnesota Educational Computing Consortium (MECC) is to assist users and educational member systems in the coordination and utilization of computing resources through cooperative planning and decision making, and to provide current computing methods and materials.

SERVICES: All MECC activities in the area of instructional computing are the responsibility of the Director of Instructional Services (Telephone: 612/376-1105). Questions related to MECC policy, procedures, or regulations should be directed to this office. The MECC Instructional Services Division is organized as follows:

Instructional Systems Development - This group is responsible for the production, coordination, refinement, and distribution of MECC instructional computing courseware products, computer programs, and their related user support material. Questions on operations within this area should be directed to the Manager, Instructional Systems Development (Telephone: 612/376-1103).

Technical Services - This group is responsible for operation and operating systems maintenance of the MECC Timeshare System (MTS), a 400+ port, all purpose, multiple language computer, which serves all Minnesota public higher education institutions and 340 school districts. Microcomputer programming and system utilities are handled by this group. Questions on operations within this area should be directed to the Manager, Technical Services (Telephone: 612/376-1141).

User Services - This group is responsible for timeshare and microcomputer user communications and training, the distribution of computing equipment, and the establishment and maintenance of the MTS telecommunications network. A staff of instructional computing coordinators are located throughout Minnesota for the purpose of promoting and facilitating computer usage. All questions on operations in this area should be directed to the Manager, User Services (Telephone: 612/376-1101).

GENERAL INFORMATION: The above information is to assist individuals who wish to contact the MECC office with specific questions. All written requests for information should be addressed to the appropriate office at MECC, 2520 BROADWAY DRIVE, ST. PAUL, MN 55113. THE FOLLOWING TWO ITEMS ADDRESS MANY ROUTINE QUESTIONS:

MECC PUBLICATIONS AND PROGRAMS PRICE LIST

No Cost. Distributed upon request. Suggest obtaining on a quarterly basis. Contact: MECC Publication's Office (Telephone: 612/376-1118).

MECC USERS Newsletters

No Cost. Distributed regularly during school year to individuals on mailing list. Contact: User Services Office (Telephone: 612/376-1117).

All requests for visitations to MECC must be scheduled in advance by calling 612/376-1130.

EVALUATION SHEET

MECC encourages your comments on this manual and the accompanying diskette. User suggestions will be carefully considered and incorporated in future documentation whenever practical.

COMMENTS ON COMPUTER PROGRAM

Diskette Name _____
Volume No. _____ Version _____
Program Name _____

COMMENTS ON MANUAL

Title of Manual _____
Program Name _____
Page No. _____

From: Name _____
Institution _____
Address _____
_____ Zip Code _____

Please detach and mail to MECC